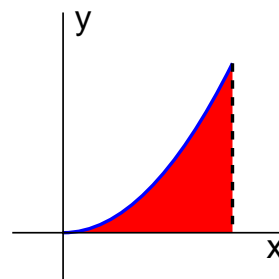


Work all seven problems. These are essay questions. To obtain maximal credit, show your work and explain your reasoning.

1. A curve on the exam Consider the curve $y = x^2$ for $0 \leq x \leq 1$ and also the region in the first quadrant of the x - y plane that lies below the curve, as indicated in the figure. For each of the following geometrically defined quantities, write an integral that represents the quantity (but do not evaluate the integral).



- the area of the region
- the volume of the horn-shaped solid obtained by revolving the region about the x -axis
- the arc length of the curve
- the surface area of the bowl-shaped surface obtained by revolving the curve about the y -axis

2. Sinc or swim The sinc function, which arises in signal processing, is defined by $\text{sinc}(x) = \frac{\sin(x)}{x}$ for $x \neq 0$. (In differential equations, the function is known as j_0 , the spherical Bessel function of the first kind of order zero.)

- How should the sinc function be defined at $x = 0$ in order to make it a continuous function?
- Find the Maclaurin series for $\text{sinc}(x)$. What is the radius of convergence of the series?
- Show that the improper integral $\int_0^\infty (\text{sinc}(x))^2 dx$ converges. (The exact value of this integral is $\pi/2$, but to prove that requires a method beyond the scope of this course: namely, the technique of contour integration in the complex plane.)

3. How hard can it be? Write down the hardest indefinite integral you can think of that you know how to evaluate by hand. Explain: why is it hard, and how would you go about evaluating it? (Do not give all the details of how to evaluate the integral, but indicate what method(s) would be needed.)

4. **The million dollar man** You know the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ by the

name “ p -series”, but it has another name: when viewed as a function of the exponent p , the series is the *zeta function* $\zeta(p)$. In the nineteenth century, the German mathematician Georg Friedrich Bernhard Riemann (1826–1866) showed how to extend this function of p to complex numbers. Based on some numerical evidence, he made a conjecture, now known as “the Riemann hypothesis”, about the location of the zeroes of the extended function. His conjecture remains unproved, and whoever succeeds in proving (or disproving) it will win one million dollars from the Clay Mathematics Institute (see http://www.claymath.org/Millennium_Prize_Problems/).



Your task today is more limited. Give an example of a positive number p that satisfies all of the following three properties simultaneously.

- (a) The infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.
- (b) The sum of the first ten thousand terms of the series approximates the whole sum with an error less than 0.01.
- (c) The sum of the first one thousand terms of the series approximates the whole sum with an error greater than 0.01.

5. **Talking trash** This is a problem about designing a container (a trash can, for example). The constraints are that the container should have a flat circular bottom, it should have no top, the sides of the container should form a surface of revolution, and the volume of the container should be π cubic units.

- (a) If the container is a cylinder, how should the radius and the height be chosen in order to minimize the amount of material used to make the container (that is, to minimize the combined area of the bottom and the sides)?
- (b) Is the cylinder in part (a) optimal, or can you design a container of a different shape (still satisfying all the constraints) that has smaller surface area than the cylinder?

6. Going off on a tangent

- (a) In three-dimensional
- x
-
- y
-
- z
- space, the equation

$$x^2 + y^2 + z^2 + 2x - 4y - 8z = 0 \quad (*)$$

defines a surface. By using the method of completing the square, show that this surface is a sphere, and determine the center and the radius.

- (b) There is a notion of a vector being tangent to a surface. The intuitive idea is that if one moves in the direction of a tangent vector, the value of the function defining the surface does not change to first order.

The precise definition is that if a surface is defined by an equation of the form $f(x, y, z) = 0$, and if $f(0, 0, 0) = 0$ (which means that the origin is a point on the surface), then a vector $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is tangent to the surface at the origin when the value of the derivative $\frac{d}{dt}(f(at, bt, ct))$ at $t = 0$ is equal to 0.

Show that the vectors $2\mathbf{i} + \mathbf{j} + 0\mathbf{k}$ and $4\mathbf{i} + 0\mathbf{j} + \mathbf{k}$ both are tangent at the origin to the sphere defined above by equation (*).

- (c) The collection of all tangent vectors to a surface at a point forms a plane, the so-called
- tangent plane*
- to the surface at the point. Find an equation for the tangent plane to the sphere (*) at the origin.

7. None of the above What is your favorite topic from this course that is not covered in the problems above? Make up an interesting problem on that topic, and then solve the problem.

HAPPY HOLIDAYS!