Sample examination problems

Knowledge is useful only if it can be applied. To see if you can apply your knowledge of calculus, I will pose some examination questions that are different from problems you have done for homework. I hope that the examinations will not only test your understanding but also expand your horizons.

Here are two sample questions that I considered putting on the first examination (but I actually chose other questions instead).

1. The *Legendre polynomials* are a special sequence of functions that arise in mechanics, electromagnetic theory, and so forth. Some of the properties of the kth Legendre polynomial $P_k(x)$ are the following.

   - $P_k(x)$ is a polynomial of degree k.
   - $P_k(1) = 1$.
   - $\displaystyle\int_{-1}^{1} [P_k(x)]^2\,dx = \frac{2}{2k+1}$.
   - Whenever $k \neq n$, the Legendre polynomials P_k and P_n satisfy the *orthogonality relation*

   $$\int_{-1}^{1} P_k(x)P_n(x)\,dx = 0.$$

   Given that $P_0(x) = 1$ and $P_1(x) = x$, determine the polynomial $P_2(x)$.

2. (a) Give an example of a rational function $r(x)$ such that the improper integral $\displaystyle\int_0^\infty r(x)\,dx$ converges.

   (b) Give an example of a rational function $r(x)$ such that the improper integral $\displaystyle\int_0^\infty r(x)\,dx$ diverges.

   (c) In general, what conditions on a rational function r guarantee that $\displaystyle\int_0^\infty r(x)\,dx$ converges?