Limits: how close is close enough?

Example

If $f(x) = 2|x|$, then $\lim_{x \to 0} f(x) = 0$.

A quantitative version of this limit is the question:
How close must x be to 0 to ensure that $f(x)$ is less than $\frac{1}{10}$?

More generally, how close must x be to 0 to ensure that $f(x)$ is less than a prescribed error tolerance ε?
Answer: $-\varepsilon/2 < x < \varepsilon/2$

The precise definition of limit

The meaning of "$\lim_{x \to b^+} f(x) = L$" is that for every positive error tolerance ε, there is a corresponding positive number δ having the following property:

$$L - \varepsilon < f(x) < L + \varepsilon \qquad \text{whenever} \qquad b < x < b + \delta.$$

For the left-hand limit $\lim_{x \to b^-} f(x)$, the corresponding property is:

$$L - \varepsilon < f(x) < L + \varepsilon \qquad \text{whenever} \qquad b - \delta < x < b.$$

For a two-sided limit $\lim_{x \to b} f(x)$, you can rewrite the inequalities using absolute value:

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - b| < \delta.$$

Assignment (not to hand in)

- In Section 2.4, Exercises 1, 3, 13, 15, 41.