Follow-up from yesterday

1. Suppose the curve $y = x^4 + ax^3 + bx^2 + cx + d$ has a tangent line when $x = 0$ with equation $y = 2x + 1$, and a tangent line when $x = 1$ with equation $y = 2 - 3x$. Find the values of a, b, c, and d.
   [#76 in 3.1. Answer: $a = 1$, $b = -6$, $c = 2$, $d = 1$.]

Strategy: There are four pieces of information to use to find the four unknowns in $f(x) = x^4 + ax^3 + bx^2 + cx + d$.

(i) $d = f(0) = 1$ (the height of the tangent line when $x = 0$).

(ii) $c = f'(0) = 2$ (the slope of the tangent line when $x = 0$).

(iii) $1 + a + b + c + d = f(1) = -1$ (the height of the tangent line when $x = 1$).

(iv) $4 + 3a + 2b + c = f'(1) = -3$ (the slope of the tangent line when $x = 1$).

Continuation

2. If $f(2) = 10$ and $f'(x) = x^2 f(x)$ for all x, find $f''(2)$.
   [#48 in 3.2. Answer: 200.]

Strategy: Apply the product rule to differentiate $x^2 f(x)$.

$f''(x) = 2x\,f(x) + x^2\,f'(x)$.

Now plug in $x = 2$.

Continuation

3. $\lim_{x\to 0} \dfrac{\sin(3x)\sin(5x)}{x^2} = ?$ [#44 in 3.3. Answer: 15.]

Strategy: Start with a simpler problem.

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\sin(x)-\sin(0)}{x-0} = \sin'(0) = \cos(0) = 1.$$

Method 1. Similarly, $\lim_{x\to 0} \dfrac{\sin(3x)}{x}$ equals the derivative of $\sin(3x)$ at $x = 0$, or $3\cos(0)$ by the chain rule.

Method 2. Rewrite $\lim_{x\to 0} \dfrac{\sin(3x)}{x} = \lim_{x\to 0} 3\dfrac{\sin(3x)}{3x} \overset{k=3x}{=} \lim_{k\to 0} 3\dfrac{\sin(k)}{k}$.

Either method shows $\lim_{x\to 0} \dfrac{\sin(3x)}{x} = 3$. Similarly $\lim_{x\to 0} \dfrac{\sin(5x)}{x} = 5$.

The chain rule

The derivative of $\sin(3x)$ equals $3\cos(3x)$.

The derivative of $\sin(5x)$ equals $5\cos(5x)$.

What about the derivative of $\sin(x^2)$?
This problem is different because the slope of the graph of x^2 is not constant.
But the same principle applies to show that the derivative equals $\cos(x^2)$ times the slope (or derivative) of x^2.

Thus $\frac{d}{dx}\sin(x^2) = \cos(x^2) \cdot 2x$

More elaborate chain rule

In general, $\dfrac{d}{dx}\sin\boxed{} = \cos\boxed{} \cdot \dfrac{d}{dx}\boxed{}$

Examples

- $\dfrac{d}{dx}\sin\left(x^2 + e^x\right) = \cos\left(x^2 + e^x\right) \cdot \left(2x + e^x\right)$
- $\dfrac{d}{dx}\sin\left(e^{x^2}\right) = \cos\left(e^{x^2}\right) \cdot e^{x^2} \cdot 2x$

Implicit differentiation

Example

Find an equation of the line tangent to the graph of $y^3 - 2xy + x^2 = 5$ at the point $(1,2)$.

Solution

Although you lack an explicit formula $y = f(x)$, you can still view the equation as $f(x)^3 - 2x\,f(x) + x^2 = 5$.

Differentiate using the chain rule and the product rule to get $3f(x)^2 f'(x) - 2f(x) - 2x\,f'(x) + 2x = 0$.

Now plug in $x = 1$ and $f(x) = y = 2$ to get $12f'(1) - 4 - 2f'(1) + 2 = 0$, so the slope $f'(1)$ equals $\frac{1}{5}$.

Therefore an equation of the tangent line is $y - 2 = \frac{1}{5}(x - 1)$.

Assignment (not to hand in)

- Section 3.4, Exercises 5, 11, 13, 21, 23, 31, 53, 63.
- Section 3.5, Exercises 3, 7, 11, 15, 27.