Examination 1

1. Suppose $\vec{v} = \langle 0, 1\rangle$ and $\vec{w} = \langle 2, 3\rangle$. Compute the length $|\vec{v} - 3\vec{w}|$.

   Solution. Since $\vec{v} - 3\vec{w} = \langle -6, -8\rangle$, the length $|\vec{v} - 3\vec{w}|$ equals $\sqrt{36+64}$, or $\sqrt{100}$, or 10.

2. Suppose vector $\vec{v}$ has length 2, and vector $\vec{w}$ has length 14, and the angle between vectors $\vec{v}$ and $\vec{w}$ is $\pi/3$ radians (equivalently, 60 degrees). Determine the dot product $\vec{v}\cdot\vec{w}$.

   Solution. The dot product of two vectors equals the product of the lengths times the cosine of the angle between the vectors. Since $\cos(\pi/3) = 1/2$, the dot product $\vec{v}\cdot\vec{w}$ equals $2\times 14\times(1/2)$, or 14.

3. Compute the vector projection of the vector $4\vec{\imath} + 3\vec{\jmath}$ onto the vector $\vec{\imath} + 2\vec{\jmath}$.

   Solution. The vector projection of $\vec{v}$ onto $\vec{w}$ equals $\left(\vec{v}\cdot\dfrac{\vec{w}}{|\vec{w}|}\right)\dfrac{\vec{w}}{|\vec{w}|}$, or $\dfrac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$. Here $\vec{v} = 4\vec{\imath}+3\vec{\jmath}$ and $\vec{w} = \vec{\imath}+2\vec{\jmath}$. Since $\vec{v}\cdot\vec{w} = (4\times 1)+(3\times 2) = 10$, and $|\vec{w}|^2 = 1^2+2^2 = 5$, the required vector projection is $\dfrac{10}{5}\vec{w}$, or $2\vec{w}$, or $2\vec{\imath}+4\vec{\jmath}$.

4. Compute the following limit: $\displaystyle\lim_{x\to 2}\frac{x^2-4}{x^2+2x-2}$.

   Solution. The indicated function is continuous when $x = 2$, so the limit is correctly obtained by substituting 2 for x:

   $$\lim_{x\to 2}\frac{x^2-4}{x^2+2x-2} = \frac{2^2-4}{2^2+2\times 2-2} = \frac{0}{6} = 0.$$

5. (a) State the precise definition of: "$\displaystyle\lim_{x\to 2} f(x) = 3$." Begin your statement as follows: "For every positive number ε, …".

   Solution. For every positive number ε, there exists a positive number δ such that $|f(x)-3| < \varepsilon$ whenever $0 < |x-2| < \delta$.

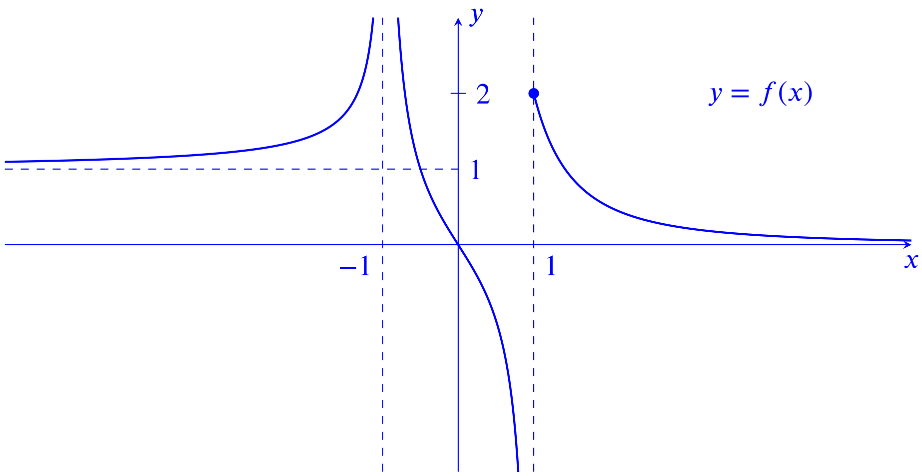
   (b) Use this definition to prove that $\displaystyle\lim_{x\to 2}(5-x) = 3$.

   Solution. Take δ to be equal to ε. To confirm that the definition is satisfied, observe that if $|x-2| < \delta = \varepsilon$, then $|f(x)-3| = |(5-x)-3| = |2-x| = |x-2| < \varepsilon$.

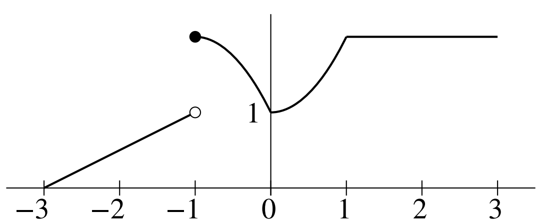
Examination 1

6. Sketch the graph of a function satisfying all of the following properties: $\lim_{x\to-\infty} f(x) = 1$, $\lim_{x\to-1} f(x) = \infty$, $f(0) = 0$, $\lim_{x\to 1^-} f(x) = -\infty$, $\lim_{x\to 1^+} f(x) = 2$, the function f is continuous from the right at 1, and $\lim_{x\to\infty} f(x) = 0$.

Solution. The following graph shows one of many possible correct answers.



7. Consider the graph of the function f shown below.

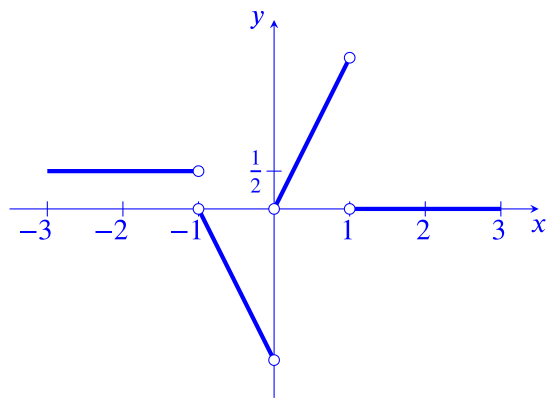


(a) At which numbers between -3 and 3 is the function *not* differentiable?

Solution. At -1, the function is not even continuous, so the function certainly is not differentiable. At 0, the slope on the left-hand side is negative but the slope on the right-hand side is approaching 0, so the function is not differentiable. At 1, the slope on the left-hand side is positive but the slope on the right-hand side is 0, so the function is not differentiable.

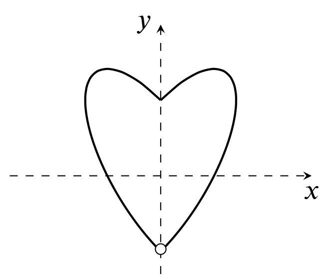
(b) Sketch the graph of f' (that is, the derivative of f).

Solution. Between -3 and -1, the slope is constantly equal to $1/2$. Between -1 and 0, the slope is negative and decreasing (getting more negative). Between 0 and 1, the slope is positive and increasing. Between 1 and 3, the slope is constantly equal to 0. The graph of the derivative therefore looks something like the graph below.



8. **Optional extra-credit problem for Valentine's Day**

The graph below is represented by parametric equations: $x = \dfrac{2t}{1+t^2}$ and $y = \dfrac{1+2|t|-t^2}{1+t^2}$ (the parameter t being an unrestricted real number).



Find the coordinates of the points on the graph at which the tangent line is vertical.

Solution. If you think of sweeping a vertical line across the picture, it should be geometrically evident that this vertical line will be tangent to the curve at the point where x takes the largest possible value. The question then becomes to determine the maximum value of the fraction $\dfrac{2t}{1+t^2}$. Since a square can never be negative, $0 \le (1-t)^2 = 1 - 2t + t^2$. Adding $2t$ to both sides shows that $2t \le 1 + t^2$, or $\dfrac{2t}{1+t^2} \le 1$. This maximum value of 1 is attained when (and only when) $t = 1$. The value of y when $t = 1$ is 1 as well, so the tangent line is vertical at the point $(1, 1)$ on the graph. By symmetry, the tangent line is vertical at the point $(-1, 1)$ too.

Another way to think about the problem is that the tangent line is vertical when the instantaneous rate of change of x equals 0, that is, when $\dfrac{dx}{dt} = 0$. Computing $\dfrac{dx}{dt}$ will be easier after we know the quotient rule for derivatives, but the calculation is feasible from the limit definition that we know now. Namely,

$$\frac{dx}{dt} = \lim_{h\to 0} \frac{\dfrac{2(t+h)}{1+(t+h)^2} - \dfrac{2t}{1+t^2}}{h} = \lim_{h\to 0} \frac{\dfrac{2(t+h)(1+t^2) - 2t[1+(t+h)^2]}{[1+(t+h)^2](1+t^2)}}{h}.$$

Now do some algebra. Combine the two denominators and multiply out the numerator to get

$$\lim_{h\to 0} \frac{2h(1-t^2-th)}{h[1+(t+h)^2](1+t^2)}.$$

Cancel the common factor of h and then send h to 0 to deduce that

$$\frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}.$$

Evidently $\dfrac{dx}{dt} = 0$ when $t = \pm 1$, which shows again that the tangent line is vertical at the points $(\pm 1, 1)$ on the graph.