Examination 2

Instructions. Your solution to each problem should include at least one complete sentence. If you make a computation, please state your strategy. (For example: "Now I calculate the first derivative by applying the quotient rule.")

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x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

Table of values for Problems 1 and 2

1. Suppose $h(x) = f(g(x))$. Use the table above to determine $h'(2)$.

   Solution. The chain rule implies that $h'(2) = f'(g(2))g'(2)$. The table says that $g(2) = 4$, so $h'(2) = f'(4)g'(2) = 7 \times 6 = 42$.

2. Use the table of values above to explain why there must be some real number x for which $f''(x)$ is equal to 0. Assume that the second derivative $f''(x)$ exists and is continuous for every real number x.

   Solution. *Method 1.* Since $f'(3) = 6 > 5 = f'(2)$, the second derivative f'' must be positive on some part of the interval $(2, 3)$. On the other hand, the value of f' cannot stay larger than 5 on the whole interval $(2, 3)$, for then $f(3)$ would exceed $f(2)$ by at least 5, contrary to the data in the table. Since $f'(x)$ is sometimes less than $f'(2)$ for x between 2 and 3, the second derivative f'' must be sometimes negative. Thus f'' is sometimes positive and sometimes negative, so the intermediate-value theorem implies that f'' is equal to 0 somewhere.

   Method 2. The mean-value theorem implies that there is some number c_1 between 2 and 3 for which $\dfrac{f(3) - f(2)}{3 - 2} = f'(c_1)$, that is, $1 = f'(c_1)$. Similarly, there is a number c_2 between 3 and 4 for which $\dfrac{f(4) - f(3)}{4 - 3} = f'(c_2)$, that is, $1 = f'(c_2)$. Since f' has equal values at c_1 and at c_2, Rolle's theorem implies that there is some number c_3 between c_1 and c_2 for which $f''(c_3) = 0$.

3. Find the slope of the curve $x^{42} + xy + y^3 = 1$ at the point on the curve where $x = 1$.

Solution. First observe that when $x = 1$, the equation says that $1 + y + y^3 = 1$, or $y(1 + y^2) = 0$, so $y = 0$. Thus the relevant point on the curve is the point $(1, 0)$.

Now apply the method of implicit differentiation, invoking the product rule and the chain rule, to see that

$$42x^{41} + xy' + y + 3y^2y' = 0.$$

When $x = 1$ and $y = 0$, this equation says that $42 + y' + 0 + 0 = 0$, so $y' = -42$. Thus the slope of the curve at the indicated point is equal to -42.

4. The parametric equations $x = t^3 - 2t$ and $y = 10t^3 + 6t^2$ determine a curve. Find an equation for the line tangent to the curve at the point on the curve where $t = 1$.

Solution. The tangent vector $\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$ equals $\langle 3t^2 - 2, 30t^2 + 12t\rangle$, and when $t = 1$ this vector becomes $\langle 1, 42\rangle$. The corresponding slope is 42.

When $t = 1$, the value of x is -1, and the value of y is 16. By the point-slope formula, the equation of the tangent line is $y - 16 = 42(x + 1)$, or $y = 42x + 58$.

5. Determine the maximum value of the polynomial $x^4 - 4x^3 + 4x^2 + 41$ on the interval where $0 \le x \le 2$.

Solution. First determine the critical numbers by finding the numbers at which the derivative equals 0:

$$4x^3 - 12x^2 + 8x = 0 \quad\iff\quad 4x(x^2 - 3x + 2) = 0 \quad\iff\quad 4x(x - 1)(x - 2) = 0.$$

Therefore the critical numbers are 0, 1, and 2. Two of these numbers are endpoints of the interval, and the remaining one is an interior point of the interval.

Now evaluate the original polynomial at each of the three numbers. When $x = 0$, the value of the polynomial is 41. When $x = 1$, the value of the polynomial is $1-4+4+41$, or 42. When $x = 2$, the value of the polynomial is $16 - 32 + 16 + 41$, or 41. The conclusion is that the minimum value of the polynomial on the given interval is 41, and the maximum value is 42.

Remark. The problem actually can be solved without calculus. Observe that

$$x^4 - 4x^3 + 4x^2 + 41 = x^2(x^2 - 4x + 4) + 41 = x^2(x - 2)^2 + 41.$$

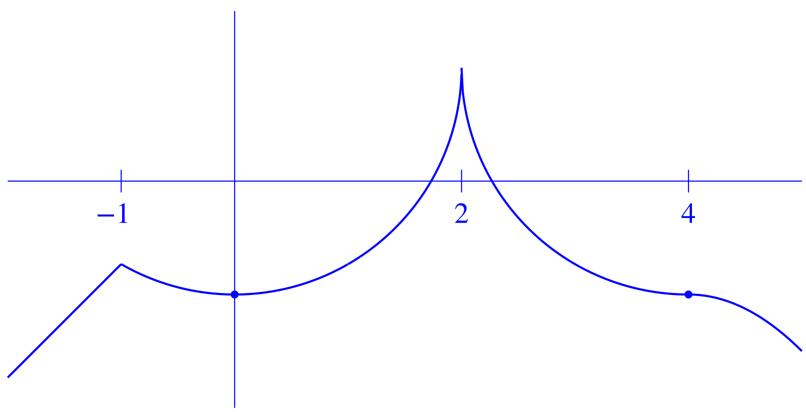
To make the problem more symmetric, introduce a new variable u such that $x = u + 1$. When x lies between 0 and 2, the variable u lies between -1 and 1. In terms of the new variable, the polynomial has the form

$$(u + 1)^2(u - 1)^2 + 41 \qquad\text{or}\qquad (u^2 - 1)^2 + 41 \qquad\text{or}\qquad (1 - u^2)^2 + 41.$$

When u is between -1 and 1, the quantity u^2 is between 0 and 1, so $1 - u^2$ is between 0 and 1, and $(1 - u^2)^2$ is between 0 and 1. Therefore $(1 - u^2)^2 + 41$ is between 41 and 42. The maximum value of 42 is attained when $u = 0$, that is, when $x = 1$.

6. Sketch the graph of a function f satisfying all of the following properties: $f'(x) = 1$ when $x < -1$; $f'(x) < 0$ when $-1 < x < 0$; $f'(0) = 0$; $f'(x) > 0$ when $0 < x < 2$; $\lim_{x\to 2^-} f'(x) = \infty$; $\lim_{x\to 2^+} f'(x) = -\infty$; $f'(x) < 0$ when $2 < x < 4$; $f'(4) = 0$; and $f'(x) < 0$ when $x > 4$.

Solution. The indicated information shows that the graph has slope 1 when $x < -1$; the graph is decreasing when $-1 < x < 0$; there is a local minimum when $x = 0$; the graph is increasing when $0 < x < 2$; when $x = 2$ there is either a vertical asymptote or a vertical tangent line; the graph is decreasing when $2 < x < 4$; there is a saddle point (neither a local maximum nor a local minimum) when $x = 4$; and the graph is decreasing when $x > 4$. The graph below is one possible picture. This problem is a simplification of Exercise 30 in Section 4.3 of the textbook.



7. When x is a small positive number, is e^{-42x} larger than $1 - 42x$ or smaller than $1 - 42x$? Explain how you know.

Solution. *Method 1.* If $f(x) = e^{-42x}$, then $f(0) = e^0 = 1$. By the chain rule, $f'(x) = -42e^{-42x}$, so $f'(0) = -42$. Accordingly, the expression $1 - 42x$ can be interpreted either as the linear approximation to e^{-42x} at 0 or the equation of the tangent line to the graph of f at the point $(0, 1)$ on the graph.

Since $f''(x) = (-42)^2 e^{-42x}$, the second derivative is always positive. Accordingly, the graph of f is concave up and lies above the tangent line. Therefore $e^{-42x} > 1 - 42x$ when $x \neq 0$. The smallness of x is irrelevant: the graph lies above the tangent line for every x, small or large.

Method 2. Let $g(x)$ denote the difference $e^{-42x} - (1 - 42x)$. Evidently $g(0) = 0$. Now

$$g'(x) = -42e^{-42x} + 42 = 42(1 - e^{-42x}).$$

When x is positive, the decaying exponential e^{-42x} is smaller than 1, so $g'(x)$ is positive. Therefore $g(x)$ is increasing for positive x. Since $g(0) = 0$, the increasing function $g(x)$ is greater than 0 when x is positive. In other words, e^{-42x} is larger than $1 - 42x$ when x is positive. (A similar argument shows that e^{-42x} is larger than $1-42x$ also when x is negative.)

8. *Optional extra-credit problem for March Madness.*
Suppose the volume of a sphere is increasing at a rate of $(48/7)$ cm^3/sec. How fast is the circumference of the sphere changing when the radius is 12 cm?

Remark. This problem is motivated by the current NCAA basketball tournament, in which the TAMU women's team has advanced to the third round. The size of a basketball is commonly stated in terms of the circumference, which equals 2π times the radius. The volume of a sphere equals $\frac{4}{3}\pi$ times the cube of the radius. A men's basketball has a radius of about 12 cm, and a women's basketball has a radius about half a centimeter smaller.

Solution. If V denotes the volume, and C denotes the circumference, and r denotes the radius, then $V = \frac{4}{3}\pi r^3$ and $C = 2\pi r$. The chain rule implies that $\frac{dC}{dt} = 2\pi\frac{dr}{dt}$ and $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} = 2r^2 \times 2\pi\frac{dr}{dt} = 2r^2\frac{dC}{dt}$. Thus $\frac{dC}{dt} = \frac{1}{2r^2}\frac{dV}{dt}$. Inserting the given values shows that

$$\frac{dC}{dt} = \frac{1}{2(12)^2}\frac{48}{7} = \frac{1}{42}. \qquad \text{(The units are cm/sec.)}$$

Remark. When you inflate a real basketball, what you are primarily doing is increasing the pressure of the air inside, not increasing the volume.