Several Variable Calculus

Instructions These problems should be viewed as essay questions. Before making a calculation, you should explain in words what your strategy is.

Please write your solutions on your own paper. Each of the 10 problems counts for 10 points.

1. Evaluate the iterated integral $\int_0^{\pi/2}\int_0^1 x\cos(xy)\,\mathrm{d}y\,\mathrm{d}x$.

   Solution. Observe that $x\cos(xy)$ is the y-partial derivative of $\sin(xy)$. Then

$$\begin{aligned}\int_0^{\pi/2}\int_0^1 x\cos(xy)\,\mathrm{d}y\,\mathrm{d}x &= \int_0^{\pi/2} \sin(xy)\Big|_{y=0}^{y=1}\,\mathrm{d}x\\ &= \int_0^{\pi/2}(\sin(x)-0)\,\mathrm{d}x\\ &= -\cos(x)\Big|_0^{\pi/2}\\ &= -\cos(\pi/2)+\cos(0)\\ &= 1.\end{aligned}$$

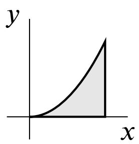
2. Describe the solid whose volume is given by the spherical-coordinate triple integral

$$\int_0^{\pi}\int_0^{\pi}\int_1^2 \rho^2\sin(\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta.$$

   (You do not need to evaluate the integral.)

   Solution. Notice that $\rho^2\sin(\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$ is $\mathrm{d}V$, the volume element, expressed in spherical coordinates. Since ρ goes from 1 to 2, the solid is contained in the shell between a sphere of radius 1 and a sphere of radius 2. The angle ϕ goes from 0 to π, its maximum range, so this angle does not restrict the solid. The angle θ goes from 0 to π, half its maximum range, so the solid lies in the half-space where $y > 0$. Thus the solid is a hemisphere of radius 2 with a concentric hemisphere of radius 1 removed.

   The problem does not ask for the value of the integral. But that value works out to $14\pi/3$.



3. Evaluate the double integral $\iint_D xe^y \, \mathrm{d}A$, where D is the region in the first quadrant bounded by the line $y = 0$, the line $x = 1$, and the parabola $y = x^2$.

Solution. The convenient method is to integrate with respect to y first:

$$\begin{aligned}
\iint_D xe^y \, \mathrm{d}A &= \int_0^1 \int_0^{x^2} xe^y \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 xe^y \Big|_{y=0}^{y=x^2} \mathrm{d}x \\
&= \int_0^1 x \left(e^{x^2} - 1\right) \mathrm{d}x \\
&\overset{u=x^2}{=} \int_0^1 \frac{1}{2} \left(e^u - 1\right) \mathrm{d}u \\
&= \frac{1}{2} \left[e^u - u\right]_0^1 \\
&= \frac{1}{2} \left[(e-1) - (1-0)\right] \\
&= \frac{e-2}{2}.
\end{aligned}$$

4. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $x + z = 2$.

Solution. One method is to use cylindrical coordinates. The variable z goes from 0 to $2 - x$, or $2 - r\cos(\theta)$. Since the equation $x^2 + y^2 = 1$ describes a circle of radius 1, the angle θ is unrestricted, and r goes from 0 to 1. Here is the iterated integral:

$$\begin{aligned}
\int_0^{2\pi} \int_0^1 \int_0^{2-r\cos(\theta)} r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta &= \int_0^{2\pi} \int_0^1 (2 - r\cos(\theta)) r \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^{2\pi} \left[r^2 - \frac{1}{3} r^3 \cos(\theta)\right]_0^1 \mathrm{d}\theta \\
&= \int_0^{2\pi} \left[1 - \frac{1}{3}\cos(\theta)\right] \mathrm{d}\theta \\
&= 2\pi.
\end{aligned}$$

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5. Rewrite the integral $\int_0^2 \int_0^y \int_0^{y^2} f(x, y, z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$ as an iterated integral in the order $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

   Solution. The solid lies in the first octant (since all three variables start at 0). The solid lies under the parabolic cylinder $z = y^2$ (with axis along the x-axis) and between the plane $x = y$ and the yz plane. Also the solid stops at the plane $y = 2$.

   Consequently, the variable x goes from 0 to y; then y goes from $\sqrt{z}$ to 2, and z goes from 0 to 4 (since the equation $z = y^2$ says that $z = 4$ when $y = 2$). Here is the iterated integral:

$$\int_0^4 \int_{\sqrt{z}}^2 \int_0^y f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

6. Set up an integral for the surface area of the part of the paraboloid $z = x^2+y^2$ that lies between the plane $z = 1$ and the plane $z = 4$.
   (You do not need to evaluate the integral.)

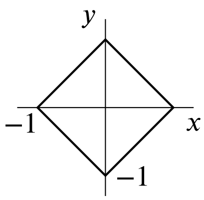
   Solution. The surface area element is

$$\begin{aligned}\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} &= \sqrt{1 + (2x)^2 + (2y)^2} \\ &= \sqrt{1 + 4(x^2 + y^2)}.\end{aligned}$$

   Since z goes from 1 to 4, the surface lies over the region in the xy-plane between the circle $1 = x^2+y^2$ and the circle $4 = x^2+y^2$ (which has radius 2). Because of the circular symmetry, it is convenient to write the integral in polar coordinates:

$$\int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2}\, r\,\mathrm{d}r\,\mathrm{d}\theta.$$

   The problem does not ask for the value of the integral. But the value works out to $\frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$.



7. Let D be the square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$. Rewrite the double integral $\iint_D (x + y)^5 \, \mathrm{d}A$ as an integral with respect to $\mathrm{d}u \, \mathrm{d}v$, where $u = x + y$ and $v = x - y$. (You do not need to evaluate the integral.)

Solution. Compute the Jacobian:

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2.$$

The area-magnification factor for the change of variables is the absolute value of the Jacobian, so $\mathrm{d}u \, \mathrm{d}v = 2 \, \mathrm{d}x \, \mathrm{d}y$, or $\mathrm{d}x \, \mathrm{d}y = \frac{1}{2} \, \mathrm{d}u \, \mathrm{d}v$.

The side of the square in the first quadrant of the xy-plane is part of the line $x + y = 1$, which turns into the equation $u = 1$. Similarly, the parallel side in the third quadrant turns into the equation $u = -1$. The side in the fourth quadrant is part of the line $x - y = 1$, which corresponds to $v = 1$, and the parallel side in the second quadrant corresponds to $v = -1$.

Consequently, the new integral in the uv-plane is

$$\int_{-1}^{1} \int_{-1}^{1} \frac{1}{2} u^5 \, \mathrm{d}u \, \mathrm{d}v.$$

The problem does not ask for the value of the integral. But the value is 0 by symmetry!

8. Find the work done by the force field $\vec{F}(x, y) = y^2 \, \hat{\imath} + 2xy \, \hat{\jmath}$ on a particle that moves in a straight line from the point $(1, 2)$ to the point $(5, 1)$.

Solution. One method is to observe that $\vec{F}(x, y) = \nabla(xy^2)$. Therefore the work integral $\int \vec{F} \cdot \mathrm{d}\vec{r}$ equals

$$xy^2 \Big|_{(1,2)}^{(5,1)} = 5 - 4 = 1.$$

Another method is to parametrize the line via

$$\begin{aligned} x &= 1 + 4t \\ y &= 2 - t \end{aligned} \qquad 0 \le t \le 1.$$

Then $\mathrm{d}x = 4\,\mathrm{d}t$ and $\mathrm{d}y = -\,\mathrm{d}t$, so

$$\begin{aligned}\int \vec{F}\cdot \mathrm{d}\vec{r} &= \int y^2\,\mathrm{d}x + 2xy\,\mathrm{d}y \\ &= \int_0^1 \left[(2-t)^2(4) + 2(1+4t)(2-t)(-1)\right]\mathrm{d}t \\ &= \int_0^1 \left(12t^2 - 30t + 12\right)\mathrm{d}t \\ &= 4 - 15 + 12 \\ &= 1.\end{aligned}$$

9. Evaluate the line integral $\int_C y\,\mathrm{d}s$ when the parametric equations of C are $x = t^2$ and $y = 2t$, where $0 \le t \le 1$.

Solution. The arc length element $\mathrm{d}s$ equals

$$\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \sqrt{\left(2t\right)^2 + 2^2}\,\mathrm{d}t = 2\sqrt{t^2+1}\,\mathrm{d}t.$$

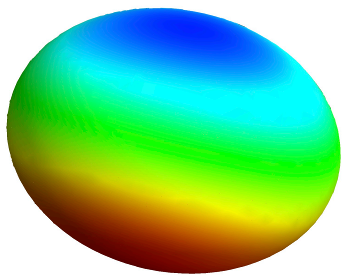
Therefore

$$\int_C y\,\mathrm{d}s = \int_0^1 4t\sqrt{t^2+1}\,\mathrm{d}t = \frac{4}{3}\left(t^2+1\right)^{3/2}\Big|_0^1 = \frac{4}{3}\left[2^{3/2}-1\right].$$

10. What does it mean to say that a vector field $\vec{F}$ is a *conservative* vector field?

Solution. A conservative vector field is the same thing as a gradient vector field: there exists a function f such that $\vec{F} = \nabla f$.

Optional bonus problem for extra credit



Find the volume of an egg whose eggshell has the equation

$$\frac{x^2}{8} + \frac{y^2}{5} + \frac{z^2}{3} = 1.$$

Solution. The problem asks for $\iiint_D \mathrm{d}V$, where D is the region inside the surface. One method is to change variables via $x = u\sqrt{8}$ and $y = v\sqrt{5}$ and $z = w\sqrt{3}$. Then $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \sqrt{120}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$, and the equation of the surface in the new coordinates becomes $u^2 + v^2 + w^2 = 1$, a sphere of radius 1. Since the volume of a ball of radius r equals $\frac{4}{3}\pi r^3$, the volume of a ball of radius 1 equals $\frac{4}{3}\pi$. Accordingly,

$$\iiint_D \mathrm{d}V = \sqrt{120}\left(\tfrac{4}{3}\pi\right), \qquad \text{or} \qquad \frac{8\pi\sqrt{30}}{3}.$$

(If you forgot the formula for the volume of a ball of radius 1, then you could compute it in spherical coordinates as

$$\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2 \sin(\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta,$$

which works out to $4\pi/3$.)