Several Variable Calculus

1. Evaluate the cross product $\vec{a} \times \vec{b}$ when $\vec{a} = \hat{\imath} + 2\hat{\jmath}$ and $\vec{b} = 3\hat{\jmath} + 4\hat{k}$.

   Solution. One method is to use the determinant rule for cross products:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 0 \\ 0 & 3 & 4 \end{vmatrix} = \hat{\imath} \begin{vmatrix} 2 & 0 \\ 3 & 4 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 8\hat{\imath} - 4\hat{\jmath} + 3\hat{k}.$$

   An alternative method is to use that the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are a right-hand orthogonal system, so $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$, and $\hat{k} \times \hat{\imath} = \hat{\jmath}$. Accordingly, distributing the multiplication shows that

$$\begin{aligned} (\hat{\imath} + 2\hat{\jmath}) \times (3\hat{\jmath} + 4\hat{k}) &= 3(\hat{\imath} \times \hat{\jmath}) + 4(\hat{\imath} \times \hat{k}) + 6(\hat{\jmath} \times \hat{\jmath}) + 8(\hat{\jmath} \times \hat{k}) \\ &= 3\hat{k} - 4\hat{\jmath} + 8\hat{\imath} \end{aligned}$$

   (since $\hat{\jmath} \times \hat{\jmath} = 0$, and $\hat{\imath} \times \hat{k} = -\hat{k} \times \hat{\imath}$).

2. Write a sentence stating a method for determining the angle between the vectors $\langle 1, 2, 3 \rangle$ and $\langle 1, 1, -1 \rangle$. Then carry out your strategy, and find the angle.

   Solution. The usual strategy is to use that the dot product of two vectors is the product of their lengths times the cosine of the angle between them, so computing the lengths and the dot product makes it possible to solve for the cosine and hence for the angle. In the case at hand,

$$\langle 1, 2, 3 \rangle \cdot \langle 1, 1, -1 \rangle = 1 + 2 - 3 = 0.$$

   Therefore the two vectors are orthogonal: the cosine of the angle equals 0, so the angle is $\pi/2$ radians (or 90°).

3. State a method for determining a unit vector that is orthogonal (perpendicular) to the plane passing through the three points $(1, 0, 0)$, $(1, 2, 0)$, and $(1, 0, 3)$. Then carry out your strategy, and find the vector.

   Solution. The usual method is to generate two vectors in the plane by subtracting the coordinates of pairs of points in the plane; then take the cross product of the vectors to produce a vector orthogonal to the plane.

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The vector joining the point $(1,0,0)$ to the point $(1,2,0)$ is $\langle 0,2,0\rangle$, or $2\hat{\jmath}$, and the vector joining the point $(1,0,0)$ to the point $(1,0,3)$ is $\langle 0,0,3\rangle$, or $3\hat{k}$. The cross product $2\hat{\jmath}\times 3\hat{k}$ equals $6\hat{\imath}$, and a corresponding unit vector is $\hat{\imath}$. If you choose the points in a different order, you could get the vector $-\hat{\imath}$, which is a correct answer too.

An alternative method is to think about the problem geometrically. Since all three of the given points have the same first coordinate, the plane that the points determine must be the plane where $x = 1$. A vector orthogonal to a plane on which x is constant must point in the direction of increasing x (or decreasing x), so the required unit vector is $\hat{\imath}$ (or $-\hat{\imath}$).