Several Variable Calculus

1. Calculate the iterated integral $\int_0^{\pi/2} \int_0^{\pi/2} \sin(x+y)\,\mathrm{d}y\,\mathrm{d}x$.

Solution. This problem is Exercise 10 on page 804 of the textbook. Here is the calculation:

$$\begin{aligned}
\int_0^{\pi/2} \left(\int_0^{\pi/2} \sin(x+y)\,\mathrm{d}y \right) \mathrm{d}x &= \int_0^{\pi/2} \Big[-\cos(x+y) \Big]_{y=0}^{y=\pi/2} \mathrm{d}x \\
&= \int_0^{\pi/2} \Big(-\cos(x+\tfrac{\pi}{2}) + \cos(x) \Big) \mathrm{d}x \\
&= \Big[-\sin(x+\tfrac{\pi}{2}) + \sin(x) \Big]_0^{\pi/2} \\
&= \Big(-\sin(\pi) + \sin(\tfrac{\pi}{2}) \Big) - \Big(-\sin(\tfrac{\pi}{2}) + \sin(0) \Big) \\
&= 2.
\end{aligned}$$

2. Calculate the double integral

$$\iint_R y\,e^{xy}\,\mathrm{d}A, \qquad \text{where } R = [0,1] \times [0,1].$$

Solution. This problem is the same as an example that we worked in class (Exercise 20 on page 804) except with the variables x and y interchanged. The effective way to solve this problem is to integrate first with respect to x:

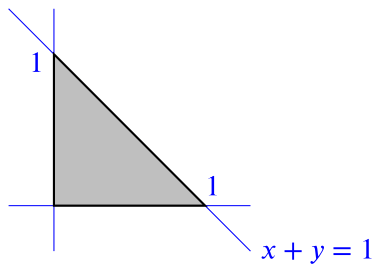
$$\begin{aligned}
\iint_R y\,e^{xy}\,\mathrm{d}A &= \int_0^1 \int_0^1 y\,e^{xy}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 \Big[e^{xy} \Big]_{x=0}^{x=1} \mathrm{d}y \\
&= \int_0^1 \Big(e^y - 1 \Big) \mathrm{d}y \\
&= \Big[e^y - y \Big]_0^1 \\
&= \Big(e - 1 \Big) - \Big(1 - 0 \Big) \\
&= e - 2.
\end{aligned}$$

Several Variable Calculus

If instead you try to integrate first with respect to y, then you have to integrate by parts, and you end up with a difficult integral in the x variable.

3. Evaluate the double integral $\iint_D y\,\mathrm{d}A$, where D is the triangular region with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

Solution. Here is a picture of the region D:



The work required to compute the iterated integral is about the same in either order. Here is the computation with the x integration first:

$$\begin{aligned}\int_0^1\int_0^{1-y} y\,\mathrm{d}x\,\mathrm{d}y &= \int_0^1 \Big[xy\Big]_{x=0}^{x=1-y}\mathrm{d}y\\ &= \int_0^1 (1-y)y\,\mathrm{d}y\\ &= \int_0^1 (y-y^2)\,\mathrm{d}y\\ &= \Big[\tfrac{1}{2}y^2-\tfrac{1}{3}y^3\Big]_0^1\\ &= \tfrac{1}{6}.\end{aligned}$$

And here is the computation with the y integration first:

$$\begin{aligned}\int_0^1\int_0^{1-x} y\,\mathrm{d}y\,\mathrm{d}x &= \int_0^1 \Big[\tfrac{1}{2}y^2\Big]_{y=0}^{y=1-x}\mathrm{d}x\\ &= \tfrac{1}{2}\int_0^1 (1-x)^2\,\mathrm{d}x\\ &= \tfrac{1}{2}(-\tfrac{1}{3})\Big[(1-x)^3\Big]_0^1 = \tfrac{1}{6}.\end{aligned}$$