Several Variable Calculus

1. Green's theorem says that

$$\int_C P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}A$$

What do the symbols C and D represent in this formula?

Solution. The statement of Green's theorem on page 892 in the textbook says that C is a piecewise-smooth, positively oriented, simple closed curve in the plane, and D is the region bounded by C.

The most essential hypothesis is that C is a *closed* curve, meaning that the curve starts and stops at the same point.

The word "simple" means that the curve does not cross itself. (If a curve crosses itself, then there is an ambiguity about which points are inside the curve and which points are outside.) A simple closed curve has an inside and an outside, and D denotes the region inside the curve C.

The "positive" orientation for C is counterclockwise. (Reversing the direction of the curve changes the sign of the integral.)

The hypothesis of piecewise smoothness (which is defined way back on page 699 in section 11.7) is a technical assumption to guarantee that the line integral makes sense. Since the definition of a line integral involves the tangent vector to the curve, the curve needs to have a well-defined tangent vector, except perhaps at a finite number of points. And the tangent vector should change in a continuous way, since continuous functions are the ones that you know can be integrated.

2. Apply Green's theorem to evaluate the line integral $\int_C \sin(x^2)\,\mathrm{d}x + x\,\mathrm{d}y$, where the closed curve C is the square with vertices $(\pm 1, \pm 1)$.

Solution. If the orientation of C is counterclockwise, then Green's theorem implies that

$$\int_C \sin(x^2)\,\mathrm{d}x + x\,\mathrm{d}y = \iint_D \left(\frac{\partial x}{\partial x} - \frac{\partial \sin(x^2)}{\partial y}\right)\mathrm{d}A = \iint_D 1\,\mathrm{d}A.$$

In other words, the integral equals the area of a square whose sides have length 2. This area is 4.

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3. Find the work done by the force $\vec{F}(x, y) = -y\hat{\imath} + x\hat{\jmath}$ in moving a particle once around the circle with equation $x^2 + y^2 = 1$.

Solution. The work is $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$, or $\int_C -y\,\mathrm{d}x + x\,\mathrm{d}y$. Green's theorem implies that if the circle is traversed in the counterclockwise direction, then

$$\int_C -y\,\mathrm{d}x + x\,\mathrm{d}y = \iint_D \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right) \mathrm{d}A = \iint_D 2\,\mathrm{d}A.$$

In other words, the work equals twice the area of a circle of radius 1. Thus the work equals 2π.

An alternative method (without using Green's theorem) is to parametrize the circle by $x = \cos(\theta)$ and $y = \sin(\theta)$. Then

$$\begin{aligned}\int_C -y\,\mathrm{d}x + x\,\mathrm{d}y &= \int_0^{2\pi} \big(-\sin(\theta)(-\sin(\theta)) + \cos(\theta)\cos(\theta) \big)\,\mathrm{d}\theta \\ &= \int_0^{2\pi} \left(\sin^2(\theta) + \cos^2(\theta) \right) \mathrm{d}\theta \\ &= \int_0^{2\pi} 1\,\mathrm{d}\theta \\ &= 2\pi.\end{aligned}$$