Several Variable Calculus

1. The final examination is on what date and at what time?

   Solution. The final examination takes place on the third of May from 15:00 to 17:00, or, more memorably, 3–5 on 5/3.

2. Find curl $\vec{F}$ and div $\vec{F}$ [that is, $\nabla \times \vec{F}$ and $\nabla \cdot \vec{F}$] when

$$\vec{F}(x, y, z) = (\sin x)\hat{\imath} + (\cos x)\hat{\jmath} + z^2\hat{k}.$$

   Solution. This problem is a routine computation:

$$\begin{aligned}
\nabla \times \vec{F} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin x & \cos x & z^2 \end{vmatrix} \\
&= \hat{\imath} \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \cos x & z^2 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ \sin x & z^2 \end{vmatrix} + \hat{k} \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ \sin x & \cos x \end{vmatrix} \\
&= -(\sin x)\hat{k},
\end{aligned}$$

   and

$$\begin{aligned}
\nabla \cdot \vec{F} &= \frac{\partial}{\partial x}(\sin x) + \frac{\partial}{\partial y}(\cos x) + \frac{\partial}{\partial z}(z^2) \\
&= \cos x + 2z.
\end{aligned}$$

3. Set up an integral for the surface area of the parametric surface given by

$$\vec{r}(u, v) = v^2\hat{\imath} - uv\hat{\jmath} + u^2\hat{k}, \qquad 0 \le u \le 1, \quad 0 \le v \le 2.$$

   (Do not attempt to evaluate the integral!)

   Solution. The element of surface area in parametric form is $\mathrm{d}u\,\mathrm{d}v$ times the area of the parallelogram determined by the tangent vectors $\dfrac{\partial \vec{r}}{\partial u}$ and $\dfrac{\partial \vec{r}}{\partial v}$: namely, the length of the cross product $\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}$. Here is the computation.

$$\frac{\partial \vec{r}}{\partial u} = 0\hat{\imath} - v\hat{\jmath} + 2u\hat{k} \qquad \text{and} \qquad \frac{\partial \vec{r}}{\partial v} = 2v\hat{\imath} - u\hat{\jmath} + 0\hat{k},$$

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so

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & -v & 2u \\ 2v & -u & 0 \end{vmatrix} = 2u^2\hat{\imath} + 4uv\hat{\jmath} + 2v^2\hat{k}.$$

The length of this vector is $\sqrt{4u^4 + 16u^2v^2 + 4v^4}$. Therefore the surface area is given by the integral

$$\int_0^2 \int_0^1 \sqrt{4u^4 + 16u^2v^2 + 4v^4}\, du\, dv,$$

or

$$2\int_0^2 \int_0^1 \sqrt{u^4 + 4u^2v^2 + v^4}\, du\, dv.$$

(This integral cannot be evaluated in terms of elementary functions.)