Math 304
Linear Algebra

Harold P. Boas

boas@tamu.edu

April 22, 2010

Snapshot

From last time:

- Eigenvalues and eigenvectors

Today:

- Application of eigenvectors to systems of linear differential equations

Next time:

- Diagonalization of matrices (revisited)

Reminders on first-order linear differential equations

Example. Solve the differential equation $\frac{dy}{dt} = 3y$ (or $y' = 3y$) with the initial condition $y(0) = 4$.

Solution. A function whose derivative equals 3 times the function is $y(t) = e^{3t}$, or more generally $y(t) = ce^{3t}$ for an arbitrary constant c.

Thus the *general solution* to $y' = 3y$ is $y(t) = ce^{3t}$.

The *particular solution* satisfying the initial condition $y(0) = 4$ is $y(t) = 4e^{3t}$.

Linear systems of differential equations
Example: Exercise 1(b), page 323

Find the general solution to the system $\begin{cases} y_1' = 2y_1 + 4y_2 \\ y_2' = -y_1 - 3y_2. \end{cases}$

Solution. Write $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}$. The system of differential equations can be written in matrix form as $\mathbf{y}' = A\mathbf{y}$.

If $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then $\mathbf{y}(t) = e^{\lambda t}\mathbf{v}$ is a solution of the differential equation. Since A has eigenvalues 1 and -2 with corresponding eigenvectors $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, the general solution is the superposition

$\mathbf{y}(t) = c_1 e^t \begin{bmatrix} 4 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, or $\begin{cases} y_1(t) = 4c_1 e^t + c_2 e^{-2t} \\ y_2(t) = -c_1 e^t - c_2 e^{-2t}. \end{cases}$

An initial condition for $\mathbf{y}(0)$ would let you determine c_1 and c_2.

Euler's formula

If the eigenvalues happen to be complex numbers, then the preceding method requires knowing about the complex exponential function.

The complex exponential function is related to the trigonometric functions via *Euler's formula*:

$$e^{it} = \cos(t) + i\sin(t).$$

For example, $e^{i\pi} = -1$, $e^{i\pi/4} = (1+i)/\sqrt{2}$, and $e^{(2+3i)t} = e^{2t}(\cos(3t) + i\sin(3t))$.

Differential equations with complex eigenvalues

Example: Exercise 1(d), page 323

Solve $\begin{cases} y_1' = y_1 - y_2 \\ y_2' = y_1 + y_2 \end{cases}$ or $\mathbf{y}' = A\mathbf{y}$ with $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

The characteristic equation is $\lambda^2 - 2\lambda + 2 = 0$. By the quadratic formula, $\lambda = \frac{2\pm\sqrt{4-8}}{2} = 1 \pm i$. An eigenvector corresponding to eigenvalue $1+i$ is $\begin{bmatrix} i \\ 1 \end{bmatrix}$. One complex-valued solution is

$$\mathbf{y}(t) = e^{(1+i)t}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} e^t(-\sin(t) + i\cos(t)) \\ e^t(\cos(t) + i\sin(t)) \end{bmatrix}.$$

The differential equation is real-valued, so both the real part and the imaginary part of the complex solution are real solutions. The general real solution is therefore

$$\mathbf{y}(t) = c_1 e^t \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}.$$