

Topics in Applied Mathematics I

Each of the 10 problems counts 10 points.
Show your work to obtain full credit.

1. Suppose $A = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 & 2 \\ -2 & 3 & 4 \end{pmatrix}$, and $C = \begin{pmatrix} 0 & 1 \\ 5 & 2 \\ 1 & 6 \end{pmatrix}$.
- Which of the products ABC , BCA , and CAB are defined?
 - Compute one of the indicated three products that is defined.

Since AB is a 2×3 matrix, the product ABC is defined. In fact, $ABC = \begin{pmatrix} 2 & 11 \\ 53 & 62 \end{pmatrix}$. Since BC is a 2×2 matrix, the product BCA is defined. In fact, $BCA = \begin{pmatrix} -20 & 33 \\ -37 & 84 \end{pmatrix}$. Since CA is a 3×2 matrix, the product CAB is defined. In fact, $CAB = \begin{pmatrix} -4 & 9 & 8 \\ -13 & 18 & 26 \\ -25 & 54 & 50 \end{pmatrix}$. I omit the details of the computations.

2. Calculate the determinant $\begin{vmatrix} 1 & 2 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 5 \end{vmatrix}$.

Perhaps the simplest approach is to expand on the last column to get $5 \begin{vmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 4 \end{vmatrix}$. Again expand on the last column to get $5 \times 4 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 20(3 - 2) = 20$.

3. Either express the vector $(5, 6)$ as a linear combination of the vectors $(3, 4)$, $(6, 8)$, and $(-12, -16)$ or show that it is impossible to do so.

Since $(6, 8) = 2(3, 4)$ and $(-12, -16) = -4(3, 4)$, the three vectors $(3, 4)$, $(6, 8)$, and $(-12, -16)$ all lie on the same line through the origin, and so does any linear combination of them. The vector $(5, 6)$ does not lie on this line, and so it cannot be written as a linear combination of the vectors $(3, 4)$, $(6, 8)$, and $(-12, -16)$.

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4. Consider the two lines in \mathbb{R}^3 given in parametric vector form as $(0, 1, 2) + t(3, 4, 5)$ and $(6, 7, 8) + s(9, 10, 11)$.
- Are the lines parallel? Why or why not?
 - Do the lines intersect? Why or why not?

The vectors $(3, 4, 5)$ and $(9, 10, 11)$ point in different directions, so the two lines are not parallel. Lines in \mathbb{R}^3 can be skew (not parallel but also not intersecting), so we need to compute to see if there are values of t and s for which $(0, 1, 2) + t(3, 4, 5) = (6, 7, 8) + s(9, 10, 11)$. An

equivalent equation is $t \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} - s \begin{pmatrix} 9 \\ 10 \\ 11 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}$ or $t \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} - s \begin{pmatrix} 3 \\ 10 \\ 11 \end{pmatrix} =$

$\begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix}$. Adding multiples of the first row to the second and third rows

reduces the system to $t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - s \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix}$. Dividing the second

row by -2 and then adding multiples of the second row to the other rows gives $t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - s \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. There is a solution for t and s

(namely, $t = -1$ and $s = -1$), so the lines do intersect (at the point $(-3, -3, -3)$ in \mathbb{R}^3).

5. Consider the two planes in \mathbb{R}^3 given by the equations $x + y + z = 1$ and $2x + 3y + 4z = 5$. Find an equation for a third plane such that all three planes intersect in the same line.

The two given planes intersect in a line (but we do not need the equation of that line to solve this problem). To get a third plane that intersects those two planes in the same line, we need an equation that is linearly dependent on the two given ones. One such equation would be the sum of the two given equations: $3x + 4y + 5z = 6$. Another such equation would be the difference of the two given equations: $x + 2y + 3z = 4$. There are infinitely many correct answers.

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6. Suppose $\vec{u} = 2\vec{i} + 2\vec{j} - 4\vec{k}$ and $\vec{v} = 3\vec{i} - 4\vec{k}$ and $\vec{w} = 5\vec{j} - \vec{k}$.
- Which of these three vectors has the greatest length?
 - Each pair of vectors (\vec{u} and \vec{v} ; \vec{v} and \vec{w} ; \vec{u} and \vec{w}) determines an angle between 0° and 180° . Which pair of vectors determines the biggest angle?

Since $|\vec{u}| = \sqrt{4 + 4 + 16} = \sqrt{24}$ and $|\vec{v}| = \sqrt{9 + 16} = \sqrt{25}$ and $|\vec{w}| = \sqrt{25 + 1} = \sqrt{26}$, the vector \vec{w} has the greatest length.

Now $\frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} = \frac{22}{\sqrt{24}\sqrt{25}} = \cos \theta_{uv}$ and $\frac{\vec{v} \cdot \vec{w}}{|\vec{v}| |\vec{w}|} = \frac{4}{\sqrt{25}\sqrt{26}} = \cos \theta_{vw}$ and $\frac{\vec{u} \cdot \vec{w}}{|\vec{u}| |\vec{w}|} = \frac{14}{\sqrt{24}\sqrt{26}} = \cos \theta_{uw}$. Since the denominators of the fractions are nearly equal to each other, one can see even without a calculator that $\cos \theta_{vw} < \cos \theta_{uw} < \cos \theta_{uv}$. The three cosines are positive numbers, so the angles are between 0° and 90° . The cosine function is decreasing for angles between 0° and 90° , so the smallest cosine corresponds to the largest angle. Thus the angle between vectors \vec{v} and \vec{w} is the largest.

7. The matrix $\begin{pmatrix} 1-t & 0 & 1 \\ 0 & 3-t & 0 \\ 1 & 0 & 1-t \end{pmatrix}$ is invertible for most values of t . Find all the exceptional values of t for which the matrix is *not* invertible.

We want to know the values of t for which the determinant of the matrix is equal to zero. Expanding the determinant on the second row gives $(3-t) \begin{vmatrix} 1-t & 1 \\ 1 & 1-t \end{vmatrix} = (3-t)((1-t)^2 - 1) = (3-t)(t^2 - 2t) = (3-t)t(t-2)$. So the exceptional values of t are 0, 2, and 3.

Alternatively, row reduce the matrix by interchanging the top and bottom rows and subtracting $1-t$ times the top row from the bottom row to get $\begin{pmatrix} 1 & 0 & 1-t \\ 0 & 3-t & 0 \\ 0 & 0 & 1-(1-t)^2 \end{pmatrix}$. The obstacles to row reducing further to the identity matrix are that either $3-t = 0$ or $1-(1-t)^2 = 0$, which gives the same exceptional values 3, 0, and 2.

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8. Consider the following system of three simultaneous equations for the three unknowns x , y , and z :

$$\begin{cases} x - 2y + z = -1 \\ 2x - y + 2z = 4 \\ x - y + z = b \end{cases}$$

- (a) For which value(s) of b (if any) is the system inconsistent?
 (b) For which value(s) of b (if any) does the system have a unique solution?
 (c) For which value(s) of b (if any) does the system have infinitely many solutions?

Adding multiples of the first row to the other rows gives the equivalent

system $\begin{cases} x - 2y + z = -1 \\ 3y = 6 \\ y = b + 1 \end{cases}$. Dividing the second row by 2 and

then adding multiples of the new second row to the other rows gives

$\begin{cases} x + z = 3 \\ y = 2 \\ 0 = b - 1 \end{cases}$. The third equation shows that the system is

inconsistent when $b \neq 1$. When $b = 1$, the system reduces to the pair of equations $x + z = 3$ and $y = 2$, which has infinitely many solutions (z can be chosen arbitrarily, and then x is determined in terms of z via $x = 3 - z$). There is no value of b for which the system has a unique solution.

9. Suppose $f: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ is a linear function such that $f \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

$$f \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \text{ and } f \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}.$$

- (a) Find a matrix A such that $f(\vec{x}) = A\vec{x}$ for every vector \vec{x} in \mathbb{R}^3 .
 (b) Is the function f a one-to-one function? Explain why or why not.

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Since $f \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, the second column of the matrix is $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$. Since $f \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = f \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - f \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, the first column of the matrix is $\begin{pmatrix} 1 \\ -6 \end{pmatrix}$. Since $f \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = f \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - f \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -6 \end{pmatrix}$, the third column of the matrix is $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$. So $A = \begin{pmatrix} 1 & 2 & 0 \\ -6 & 2 & 5 \end{pmatrix}$.

No linear function from \mathbb{R}^3 to \mathbb{R}^2 can be one-to-one, because every triple of vectors in \mathbb{R}^2 is linearly dependent. In the case at hand, one can find a specific non-zero vector \vec{x} such that $f(\vec{x}) = \vec{0}$ by observing that this means that the dot product of \vec{x} with each row of the matrix A should be equal to zero. The cross product of the rows of A is such a vector. So $f(10, -5, 14) = 0$. Alternatively, one could row reduce to find a solution of the equation $A\vec{x} = 0$.

10. For each of the following operators on the vector space $\mathcal{M}_{2,2}$ of 2×2 matrices, say whether or not the operator is linear, and explain why.
- $A \mapsto 2A$
 - $A \mapsto$ (reduced row echelon form of A)

Multiplication by 2 is a linear operator on $\mathcal{M}_{2,2}$ because $2(A + B) = 2A + 2B$ for all matrices A and B and $2(aA) = a(2A)$ for all scalars a and all matrices A .

The function that sends a matrix to its reduced row echelon form is not linear, because the leading entry of each row in the reduced matrix is a 1, and that property is destroyed by multiplication by a scalar. (Addition also destroys the property.)

Extra credit (up to 5 points): Why does the word “matrix” have a plural form spelled “matrices” instead of “matrixes”?

Like the words “index” (plural “indices”) and “appendix” (plural “appendices”), the word “matrix” comes from Latin and retains its Latin plural.