Topics in Applied Mathematics I

Each of the 10 problems counts 10 points.
Show your work to obtain full credit.

1. Find all the eigenvectors of the matrix $\begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

   The characteristic equation is $(4-\lambda)^3 = 0$, and the eigenvalue is $\lambda = 4$ (with multiplicity 3). The eigenvectors are the non-zero vectors in the null space of the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, in other words, all vectors of the form $(a, 0, b)$, with a and b arbitrary real numbers (not simultaneously equal to zero).

2. Consider the system of equations $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & t \\ 0 & 2 & t \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ 0 \\ 1 \end{pmatrix}$ for the three unknowns x, y, and z. For which value(s) of t, if any, does the system have a unique solution? infinitely many solutions? no solution?

   Use row operations to reduce the system to row echelon form:
   $$\begin{pmatrix} 1 & -1 & 0 & t \\ 0 & 1 & t & 0 \\ 0 & 2 & t & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & t & t \\ 0 & 1 & t & 0 \\ 0 & 0 & -t & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & t+1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -t & 1 \end{pmatrix}.$$
   If $t = 0$, then the last row corresponds to the impossible equation $0 = 1$, so there is no solution (the system is inconsistent). If $t \neq 0$, then the system has a unique solution: namely, $x = t + 1$, $y = 1$, and $z = -1/t$. There is no value of t for which this system has infinitely many solutions.

3. Either find an invertible 2×2 matrix A (whose entries are real numbers) such that
   $$A^{-1} = -A$$
   (that is, the multiplicative inverse equals the additive inverse), or explain why no such matrix can exist.

There are infinitely many such matrices. One example is $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The question did not ask for all such matrices, but here is a systematic way to find them all. Set $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We want to have

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}.$$

If $b = 0$, then comparing the elements in the upper left-hand corner shows that $a^2 = -1$, which is impossible when a is a real number. So the required matrix must have $b \neq 0$.

Comparing the elements in the upper right-hand corner shows that $ad - bc = 1$. Now comparing the elements on the diagonal shows that $a = -d$. Combining these two restrictions implies that $c = -(1+a^2)/b$. Consequently, the general form of A is $\begin{pmatrix} a & b \\ -(1+a^2)/b & -a \end{pmatrix}$, where a is an arbitrary real number, and b is an arbitrary non-zero real number.

4. In the theory of Fourier series, one uses the inner product defined by

$$\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)\,dx$$

on the vector space $C[0, 2\pi]$ of functions that are continuous on the interval $[0, 2\pi]$. Determine the angle between the two vectors (functions) $f(x) = 1$ and $g(x) = x$ with respect to this inner product.

The inner product of 1 and x is $\int_0^{2\pi} 1 \cdot x\,dx$, which equals $2\pi^2$. The norm of 1 is $\left(\int_0^{2\pi} 1^2\,dx\right)^{1/2}$, which equals $\sqrt{2\pi}$. The norm of x is $\left(\int_0^{2\pi} x^2\,dx\right)^{1/2}$, which equals $\sqrt{8\pi^3/3}$. The angle between 1 and x has cosine equal to

$$\frac{2\pi^2}{\sqrt{2\pi}\sqrt{8\pi^3/3}}, \qquad \text{or} \quad \frac{\sqrt{3}}{2}.$$

Therefore the angle is $\pi/6$ radians or $30°$.

5. Either construct a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$ represented by a 4×4 matrix with the property that the null space is equal to the image, or explain why no such transformation can exist.

Since the dimension of the null space and the dimension of the image must sum to 4, the equality of these dimensions means that both equal 2. Thus we seek a 4×4 matrix having two linearly independent columns (which form a basis for the image) such that the columns are orthogonal to the rows (so that the columns are in the null space). There are many such matrices. One example is $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

6. Consider the curve in $\mathbb{R}^3$ defined parametrically by $f(t) = (t^2, e^t, \cos(t))$. At what point(s) on the curve, if any, is the tangent line to the curve parallel to one of the three coordinate axes?

   Since $f'(t) = (2t, e^t, -\sin(t))$, and e^t is never equal to 0, the tangent line is never parallel to $(1, 0, 0)$ or to $(0, 0, 1)$. The tangent line is parallel to $(0, 1, 0)$ only when $t = 0$. The corresponding point on the curve is $f(0) = (0, 1, 1)$.

7. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} xy \\ yz \\ zx \end{pmatrix}$. Find the points at which the transformation f is locally invertible. (In other words, find the points at which the Jacobian matrix is invertible.)

   The Jacobian matrix equals $\begin{pmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{pmatrix}$. The determinant equals $2xyz$. Therefore the transformation f is locally invertible at all points (x, y, z) at which all three coordinates are different from 0.

8. So-called *parabolic coordinates* (t, u, v) are related to the usual Cartesian coordinates (x, y, z) by the following formulas.

$$\begin{aligned} x &= uv\cos(t) \\ y &= uv\sin(t) \\ z &= \tfrac{1}{2}(u^2 - v^2) \end{aligned}$$

Determine the volume element in parabolic coordinates. In other words, find a function $J(t,u,v)$ such that $dx\,dy\,dz = J(t,u,v)\,dt\,du\,dv$.

The Jacobian matrix equals $\begin{pmatrix} -uv\sin(t) & v\cos(t) & u\cos(t) \\ uv\cos(t) & v\sin(t) & u\sin(t) \\ 0 & u & -v \end{pmatrix}$. Expanding along the bottom row shows that the Jacobian determinant equals $-v(-uv^2\sin^2 t - uv^2\cos^2 t) - u(-u^2v\sin^2 t - u^2v\cos^2 t) = uv^3 + u^3v = uv(u^2+v^2)$. The function J is the absolute value of the determinant of the Jacobian matrix, or $|uv|(u^2+v^2)$.

9. Consider a cylindrical can in $\mathbb{R}^3$ whose surface S consists of three pieces: a curved side defined by $x^2+y^2=1$, $0 \le z \le 1$; a bottom defined by $x^2+y^2 \le 1$, $z=0$; and a top defined by $x^2+y^2 \le 1$, $z=1$. Let $\vec{F}(x,y,z) = z\vec{i} - y\vec{j} + x\vec{k}$. Compute the flux integral $\iint_S \vec{F}\cdot\vec{n}\,d\sigma$, where the unit normal vector $\vec{n}$ is directed outward.

   By Gauss's theorem, this flux integral equals the integral of the divergence of $\vec{F}$ over the solid can. Since the divergence of $\vec{F}$ equals $\frac{\partial z}{\partial x} - \frac{\partial y}{\partial y} + \frac{\partial x}{\partial z} = -1$, the answer is the negative of the volume of the can. The volume of a cylinder is the area of the base times the height, so the answer is $-\pi$.

10. Let S be the open surface in $\mathbb{R}^3$ defined by the parametrization $g(u,v) = (u, v, u^2+v^2)$ for $u^2+v^2 \le 1$. Let $\vec{F}(x,y,z) = z^2\vec{i} + x\vec{j} + y^3\vec{k}$. Compute the integral $\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,d\sigma$, where the unit normal vector $\vec{n}$ has an orientation with positive $\vec{k}$ component.

    By the theorem of Stokes, this integral equals the integral of the field $\vec{F}$ around the border curve C: namely, $\int_C z^2\,dx + x\,dy + y^3\,dz$. (The counterclockwise orientation of C is compatible with the indicated orientation of the surface.) On the curve C, we have $z=1$ and $dz=0$, so the problem reduces to the integral $\oint dx + x\,dy$ over a unit circle. Either by using the parametrization $x=\cos(\theta)$ and $y=\sin(\theta)$ or by applying Green's theorem, one finds that this integral equals π.

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Extra credit (up to 5 points): Supply the answers to the following jokes.

(a) According to the Sesame Street character Kermit the Frog, it isn't easy being which nineteenth-century English mathematician?

It isn't easy being Green.

(b) Why is the integral of Texas politics independent of the path?

Texas politics is a conservative field.

(c) If Stokes were to work out in the weight room at the Rec Center, which biceps exercise would he do?

curls

(d) Where in the library should you go to check out $\int_C \vec{F} \cdot \vec{t}\, ds$?

the circulation desk

(e) Why does Gauss's theorem make me think of men in scuba gear?

Gauss's theorem has to do with the *divergence* of a vector field. Lloyd Bridges and Jacques Cousteau are *diver gents*.