Math 311-102

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Recap from yesterday

- vectors
- dot product
- cross product
- work
- projection
- area and volume from vector products

New topic for today: systems of equations and matrix methods

Example

Solve the system $\begin{cases} x + 2y + z = 12 \\ 4x + 3y - 11z = 13 \\ 5x - y - 28z = -17 \end{cases}$

or equivalently $\begin{pmatrix} 1 & 2 & 1 \\ 4 & 3 & -11 \\ 5 & -1 & -28 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 13 \\ -17 \end{pmatrix}$

Strategy: reduce to a simpler system with the same solutions via reversible *elementary row operations*: namely, (i) multiply a row by a non-zero scalar, and (ii) add a multiple of a row to another row (and optionally (iii) interchange two rows).

The system is *reduced* if the leading non-zero entry in each row is 1 and if the other entries in the column of each leading entry are 0.

Reduced form

The reduced system in *echelon form* is

$\begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ 0 \end{pmatrix}$ or equivalently

$\begin{cases} x - 5z = -2 \\ y + 3z = 7 \end{cases}$

One way to write the solution is
$x = 5z - 2$, $y = -3z + 7$, z arbitrary.

Another way to write the solution is the parametric form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ 0 \end{pmatrix} + t \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$$

Interpretation

Geometric interpretation: The set of solutions is a straight line passing through the point $(-2, 7, 0)$ in the direction $(5, -3, 1)$.

Algebraic interpretation: Any multiple of the vector $(5, -3, 1)$ is a solution of the corresponding *homogeneous system*

$$\begin{pmatrix} 1 & 2 & 1 \\ 4 & 3 & -11 \\ 5 & -1 & -28 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Vector $(-2, 7, 0)$ is a *particular solution* of the original system

$$\begin{pmatrix} 1 & 2 & 1 \\ 4 & 3 & -11 \\ 5 & -1 & -28 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 13 \\ -17 \end{pmatrix}$$

The general solution of the inhomogeneous system is the sum of the general solution of the homogeneous system and any particular solution of the inhomogeneous system.

Further interpretation

Another way to write the system is

$$x \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} + y \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + z \begin{pmatrix} 1 \\ -11 \\ -28 \end{pmatrix} = \begin{pmatrix} 12 \\ 13 \\ -17 \end{pmatrix}$$

In other words, the problem amounts to writing the vector on the right-hand side as a linear combination of the column vectors in the original matrix.

The existence of infinitely many solutions indicates that the columns of the matrix are *linearly dependent*: in fact,

$$5 \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ -11 \\ -28 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

You can read off the number of linearly independent column vectors by looking at the reduced echelon form of the matrix.