Math 311-102

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Image revisited

The *image* of a linear transformation is the set of all $f(\vec{x})$ as $\vec{x}$ varies over the domain.
The image is a subspace of the range.

Example. If $f\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}1&2\\3&4\\5&6\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$, what are the domain, range, and image of f?

Answer. The domain is $\mathbb{R}^2$, the range is $\mathbb{R}^3$, and the image is the set of all linear combinations of the two columns of the matrix: namely, the plane $x-2y+z=0$.

Example. Let L be the linear operator defined on the space $\mathcal{P}$ of all polynomials by $L(p)=\int_0^x p(t)\,dt$. What is the image of L?

Answer. All polynomials with constant term equal to 0.

Null space

We saw previously that a linear transformation f is one-to-one if the only solution of the equation $f(\vec{x})=0$ is $\vec{x}=0$.

The *null space* of a linear transformation is the set of all vectors $\vec{x}$ such that $f(\vec{x})=0$.
The null space is a subspace of the domain. It measures how far a linear transformation is from being one-to-one.

Example. $f\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}1&0&1\\0&1&0\\2&3&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$. Find the null space.

Solution. Row reduce to get $\begin{pmatrix}1&0&1\\0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$.

The null space consists of all scalar multiples of the vector $\vec{\imath}-\vec{k}$. This null space is a line through the origin.