Math 311-102

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Eigenvectors and eigenvalues

A non-zero vector $\vec{v}$ is an *eigenvector* of a matrix A if $A\vec{v}$ is a scalar multiple of $\vec{v}$.

Example. If $A = \begin{pmatrix} 3 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 4 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$,

then $A\vec{v} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = 2\vec{v}$.

Thus $\vec{v}$ is an eigenvector of A with *eigenvalue* equal to 2.

How can you find eigenvectors?
The strategy is to find the eigenvalues first.

Finding eigenvalues

Given a matrix A, the goal is to find a non-zero vector $\vec{v}$ and a scalar λ such that $A\vec{v} = \lambda\vec{v}$. An equivalent equation is $(A - \lambda I)\vec{v} = 0$, where I is the identity matrix.

There can be such a non-zero vector $\vec{v}$ only if the matrix $A - \lambda I$ fails to be invertible.

The equation $\det(A - \lambda I) = 0$ determines the eigenvalues; it is called the *characteristic equation* of the matrix A.

Example. Find the eigenvalues of $\begin{pmatrix} -13 & 5 \\ -30 & 12 \end{pmatrix}$.

Solution. Solve $\det \begin{pmatrix} -13-\lambda & 5 \\ -30 & 12-\lambda \end{pmatrix} = 0$ or $(-13-\lambda)(12-\lambda) - (-150) = 0$ or $\lambda^2 + \lambda - 6 = 0$ or $(\lambda+3)(\lambda-2) = 0$. The eigenvalues are 2 and -3.

Finding eigenvectors

Continuing the example: the matrix $A = \begin{pmatrix} -13 & 5 \\ -30 & 12 \end{pmatrix}$ has eigenvalues 2 and -3. Find corresponding eigenvectors.

Solution. The eigenvector $\vec{v}$ corresponding to eigenvalue 2 satisfies $(A - 2I)\vec{v} = 0$ or $\begin{pmatrix} -15 & 5 \\ -30 & 10 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$. Row reducing gives $\begin{pmatrix} -3 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$, so $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Check: $\begin{pmatrix} -13 & 5 \\ -30 & 12 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Similarly solving $(A + 3I)\vec{v} = 0$ or $\begin{pmatrix} -10 & 5 \\ -30 & 15 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$ gives $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ as an eigenvector with eigenvalue -3.

Change of basis

The matrix $A = \begin{pmatrix} -13 & 5 \\ -30 & 12 \end{pmatrix}$ represents a linear transformation with respect to the standard basis $\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$.
What matrix represents the same transformation with respect to the basis $\{\left(\begin{smallmatrix}1\\3\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)\}$ of eigenvectors?

Answer. The diagonal matrix $D = \left(\begin{smallmatrix}2 & 0\\0 & -3\end{smallmatrix}\right)$ whose diagonal entries are the eigenvalues.

The matrix $U = \left(\begin{smallmatrix}1 & 1\\3 & 2\end{smallmatrix}\right)$ (whose columns are the eigenvectors) translates from eigenvector coordinates to standard coordinates. The matrix $U^{-1} = \left(\begin{smallmatrix}-2 & 1\\3 & -1\end{smallmatrix}\right)$ translates from standard coordinates to eigenvector coordinates.

Then $A = UDU^{-1}$ and $U^{-1}AU = D$. (Check!)

We can *diagonalize* a matrix by using a basis of eigenvectors.