Math 311-102

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Tangent approximation

For functions of one variable, the tangent approximation formula says $f(x) \approx f(a) + f'(a)(x-a)$.

Example. If $f(x) = \sqrt[3]{x}$, and $a = 1$, the formula says $\sqrt[3]{x} \approx 1 + \frac{1}{3}(x-1)$.

The multi-variable approximation formula is $f(x,y) \approx f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$.

Example. If $f(x,y) = \sin(x + y\sqrt{4+y})$, and $(a,b) = (0,0)$, then $\sin(x + y\sqrt{4+y}) \approx \sin(0) + \cos(0)(x-0) + 2\cos(0)(y-0)$ $= x + 2y$.

Approximation of vector functions

Example. To approximate $f(x,y) = \begin{pmatrix} x^2+y^2 \\ x^3+\sin(y) \\ xe^y \end{pmatrix}$ near the point (a,b), use the previous formula in each row to get

$$f(x,y) \approx f(a,b) + \begin{pmatrix} 2a(x-a) + 2b(y-b) \\ 3a^2(x-a) + \cos(b)(y-b) \\ e^b(x-a) + ae^b(y-b) \end{pmatrix}$$
$$= f(a,b) + \begin{pmatrix} 2a & 2b \\ 3a^2 & \cos(b) \\ e^b & ae^b \end{pmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix}.$$

We can think of the derivative of a vector function as being a *matrix*.

The derivative matrix

The preceding example shows that the derivative of a function $f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \\ f_3(x,y) \end{pmatrix}$ should be viewed as the matrix of partial derivatives

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{pmatrix}.$$

The rows of the matrix are the *gradients* of the component functions.

The matrix is often called the *Jacobian matrix*.

Exercise

$$f(x,y,z) = \begin{pmatrix} \frac{9}{25}\sin(x) + \frac{12}{25}\tan(y) + \frac{4}{5}z \\ \frac{12}{25}x + \frac{16}{25}\sin(y) - \frac{3}{5}ze^z \\ -\frac{4}{5}x\cos(x) + \frac{3}{5}y + yz \end{pmatrix}$$

1. Find the derivative matrix of f at $(0,0,0)$.
2. Show that the matrix is *orthogonal*: the inverse equals the transpose.
3. The matrix is a rotation matrix. Find the axis of rotation.

Answers

1. The Jacobian matrix equals

$$\left.\begin{pmatrix} \frac{9}{25}\cos(x) & \frac{12}{25}\sec^2(y) & \frac{4}{5} \\ \frac{12}{25} & \frac{16}{25}\cos(y) & -\frac{3}{5}e^z - \frac{3}{5}ze^z \\ -\frac{4}{5}\cos(x) + \frac{4}{5}x\sin(x) & \frac{3}{5} + z & y \end{pmatrix}\right|_{(0,0,0)}$$

$$= \begin{pmatrix} \frac{9}{25} & \frac{12}{25} & \frac{4}{5} \\ \frac{12}{25} & \frac{16}{25} & -\frac{3}{5} \\ -\frac{4}{5} & \frac{3}{5} & 0 \end{pmatrix}.$$

2. The columns of the matrix are orthonormal.
3. The vector $\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ is an eigenvector with eigenvalue 1.