# Math 311-102

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## Reminder

The second examination is Thursday, June 23.



## Change of variables in integrals

**Example**. $\displaystyle\int_0^1 \frac{2x}{1+x^2}\,dx = \int_1^2 \frac{du}{u} = \ln 2.$
(via the change of variable $u = 1 + x^2$).

What happens in higher dimensions?

“Change of variable” means “coordinate transformation”.

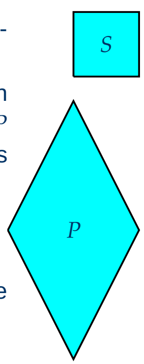
To see what happens, the place to start is with a linear transformation.



## Linear functions

**Example**. Let $S$ be the square of area $1$ with vertices at $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$ in $uv$ space.

Let $\binom{x}{y} = \left(\begin{smallmatrix} 1 & 1 \\ -2 & 2 \end{smallmatrix}\right)\binom{u}{v}$, so the image of $S$ in $xy$ space is a parallelogram $P$. The area of $P$ equals $|(\vec{\imath} - 2\vec{\jmath}) \times (\vec{\imath} + 2\vec{\jmath})|$, the length of the cross product, which equals $4$.




Thus $\int_P dx\,dy = 4 = \int_S 4\,du\,dv$.

The factor of $4$ is the determinant of the coordinate transformation $\left(\begin{smallmatrix} 1 & 1 \\ -2 & 2 \end{smallmatrix}\right)$.

**Summary**. *A linear transformation magnifies area by (the absolute value of) the determinant.*

## Jacobi's theorem

For any invertible coordinate transformation $T$ (not necessarily linear) from a region $R$ in $uv$ space to $xy$ space, $\iint_{T(R)} f(x,y)\,dx\,dy = \iint_R f(T(u,v))\,|\det T'(u,v)|\,du\,dv$ (and similarly for transformations in $\mathbb{R}^3$).

**Example** (polar coordinates). The Jacobian matrix of the coordinate transformation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$ equals $\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$, the determinant is $r$, so

$$\iint\limits_{x^2+y^2\le 1} \sqrt{x^2+y^2}\,dx\,dy = \iint\limits_{\substack{0\le r\le 1\\ 0\le\theta\le 2\pi}} \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\quad r\,dr\,d\theta$$

$$= \int_0^1 r^2\,dr \int_0^{2\pi}\,d\theta = 2\pi/3.$$



## Curvilinear coordinates

The notation for *cylindrical coordinates* in $\mathbb{R}^3$ is $(r,\theta,z)$, where $(r,\theta)$ are polar coordinates in the $xy$ plane. The volume element $dx\,dy\,dz$ transforms to $r\,dr\,d\theta\,dz$.

The notation for *spherical coordinates* in $\mathbb{R}^3$ depends on the age of the book and on the subject (mathematics or physics). The distance from a point to the origin is denoted by $r$ or $\rho$. In modern mathematics books, $\theta$ denotes the same angle as in cylindrical coordinates and $\phi$ denotes the angle measured down from the $z$-axis. Older mathematics books and many physics and engineering books reverse the meanings of $\theta$ and $\phi$.

The volume element $dx\,dy\,dz$ transforms to $r^2\sin(\text{angle down from the } z\text{-axis})\,dr\,d\theta\,d\phi$.