Math 311-102

Harold P. Boas

boas@tamu.edu

Two examples of motivating problems

1. Solve a system of linear equations, such as

$$\begin{cases} 2x_1 + 3x_2 = 7 \\ 9x_1 - 5x_2 = 4 \end{cases}$$

The mathematics involved, with thousands of variables, underlies the input-output method in economics for which Wassily Leontief won the 1973 Nobel Prize.

2. Explain electromagnetism and the propagation of light.

Maxwell's equations:
$$\begin{cases} \nabla \cdot \mathbf{E} = 4\pi\rho \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J} \end{cases}$$

Today's topic: vectors (Chapter 1)

A vector is both a *geometric* object and an *algebraic* object.

As a geometric object, a vector has a *length* and a *direction*.

Example. Find the length of the vector joining one corner of a unit cube to the opposite corner, and find the angle the vector makes with a side.

Solution. The vector $\vec{v} = (v_1, v_2, v_3)$ may be written as $(1, 1, 1)$ or $[1, 1, 1]$ or $\vec{i} + \vec{j} + \vec{k}$ or $\vec{e}_1 + \vec{e}_2 + \vec{e}_3$. By the Pythagorean theorem, the length $|\vec{v}|$ is $\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$.

A *unit* vector pointing in the same direction is $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. The vector makes equal angles with the coordinate axes, namely $\arccos \frac{1}{\sqrt{3}} \approx 54.736^\circ \approx 0.955$ radians.

Example: collision course

Problem. You are in a sailboat moving at a constant speed on a fixed heading. A sailboat on a different heading is coming closer. How can you tell if you are on a collision course?

Solution. Line up the other boat with a point on the (distant) shore. If that point doesn't move, you are on a collision course.

Mathematical explanation. Your motion along a straight line can be described parametrically as $\vec{a} + \vec{v}t$, where $\vec{a}$ is your position at time $t = 0$ and $\vec{v}$ is your velocity vector. The position of the other boat may be written similarly as $\vec{b} + \vec{w}t$. The vector pointing from your boat to the other one is the difference vector $\vec{d}(t) = (\vec{b} - \vec{a}) + (\vec{w} - \vec{v})t$. If this vector is 0 for some t, then the vectors $(\vec{b} - \vec{a})$ and $(\vec{w} - \vec{v})$ are parallel. Then $\vec{d}(t)$ points in the same direction for every t. The parallel lines in the direction $\vec{d}(t)$ "meet at infinity", that is, at a point on the distant shore.

The algebra of vectors

Vectors can be added and can be multiplied by scalars, and the operations satisfy the associative, commutative, and distributive laws.

Example. Can the vector $(607, 194, -219)$ be written as a linear combination of the vectors $(1, 2, 3)$ and $(9, 8, 7)$?

Solution. Only if we are lucky, because the vector equation $x(1, 2, 3) + y(9, 8, 7) = (607, 194, -219)$ translates to a system of *three* simultaneous equations in *two* unknowns:

$$\begin{cases} x + 9y = & 607 \\ 2x + 8y = & 194 \\ 3x + 7y = & -219 \end{cases}$$

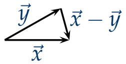
We are lucky, for $x = -311$ and $y = 102$ works in all three equations.

Dot product

By the Pythagorean theorem, the square of the length of a vector $\vec{x} = (x_1, x_2, x_3)$ is $|\vec{x}|^2 = x_1^2 + x_2^2 + x_3^2$.
This suggests introducing a *scalar product* of vectors $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ via

$$\boxed{\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3}$$

Then $\vec{x} \cdot \vec{x} = |\vec{x}|^2$.
By the law of cosines, $|\vec{x} - \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 - 2|\vec{x}|\,|\vec{y}|\,\cos(\theta)$, which simplifies to



$$\boxed{\vec{x} \cdot \vec{y} = |\vec{x}|\,|\vec{y}|\,\cos(\theta)}$$

where θ is the angle between the vectors $\vec{x}$ and $\vec{y}$.

There is nothing special about dimension 3: analogous formulas apply to the dot product of vectors in any dimension.

Projection

The *projection* of a vector $\vec{v}$ onto a vector $\vec{w}$ is

$$(\vec{v} \cdot \vec{u})\,\vec{u}, \qquad \text{where} \quad \vec{u} = \frac{\vec{w}}{|\vec{w}|}.$$

Example. A bicycle travels 3 kilometers northeast against an easterly wind that makes a resistive force of 20 newtons. How much work is done by the cyclist?

Solution. Only the component of the force $\vec{F}$ in the direction of the displacement $\vec{d}$ does work, so the work equals $\vec{F} \cdot \vec{d}$ or $20 \times 3000 \times \frac{1}{\sqrt{2}} \approx 42,426$ joules.

Cross product

The *vector product* of two three-dimensional vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ is a vector $\vec{u} \times \vec{v}$ perpendicular to both $\vec{u}$ and $\vec{v}$ with length equal to $|\vec{u}|\,|\vec{v}|\,\sin(\theta)$, where θ is the angle between $\vec{u}$ and $\vec{v}$, and with direction determined by the right-hand rule.

In particular, $\vec{i} \times \vec{j} = \vec{k} = -\vec{j} \times \vec{i}$, $\vec{j} \times \vec{k} = \vec{i} = -\vec{k} \times \vec{j}$, and $\vec{k} \times \vec{i} = \vec{j} = -\vec{i} \times \vec{k}$. The cross product is anti-commutative, so $\vec{v} \times \vec{v} = 0$ for every vector $\vec{v}$.

Example. $(2\vec{i} + 3\vec{j}) \times (4\vec{j} + 5\vec{k}) = 8(\vec{i} \times \vec{j}) + 10(\vec{i} \times \vec{k}) + 15(\vec{j} \times \vec{k})$
$= 15\vec{i} - 10\vec{j} + 8\vec{k}$.

The cross product is special to *three*-dimensional vectors.

Determinant form for cross product

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\vec{k}$$

$$= (u_2v_3 - u_3v_2)\,\vec{i} - (u_1v_3 - u_3v_1)\,\vec{j} + (u_1v_2 - u_2v_1)\,\vec{k}$$

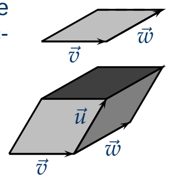
Example revisited.

$$(2\vec{i}+3\vec{j})\times(4\vec{j}+5\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 0 \\ 4 & 5 \end{vmatrix}\vec{i} - \begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix}\vec{j} + \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix}\vec{k} = 15\vec{i} - 10\vec{j} + 8\vec{k}.$$

Geometry and the cross product

The length of the cross product $|\vec{v}\times\vec{w}|$ equals the area of the parallelogram determined by the vectors $\vec{v}$ and $\vec{w}$.



The absolute value of the *scalar triple product* $|\vec{u}\cdot(\vec{v}\times\vec{w})|$ equals the volume of the *parallelepiped* determined by the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$.

The scalar triple product can be written as a determinant:

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

We will use this determinant later in Jacobi's theorem about change of variables in multiple integrals (Chapter 7).