Topics in Applied Mathematics I

1. Write a vector representation $\vec{a} + t\vec{v}$ for the line in the plane passing through the two points $(3, 11)$ and $(10, 2)$.

   The direction of the line is obtained by taking the difference of the vectors representing the two points: namely, $(10-3)\vec{i} + (2-11)\vec{j} = 7\vec{i} - 9\vec{j}$. The vector $\vec{v}$ can be any non-zero multiple of this difference. For example, $-7\vec{i} + 9\vec{j}$ is another possible choice for $\vec{v}$.

   The vector $\vec{a}$ can be taken to be any point on the line. In particular, either of the given points will serve. So two of the correct answers are $(3\vec{i} + 11\vec{j}) + t(7\vec{i} - 9\vec{j})$ and $(10\vec{i} + 2\vec{j}) + t(7\vec{i} - 9\vec{j})$.

   Other answers are possible, since neither $\vec{a}$ nor $\vec{v}$ is uniquely determined. Also, the answer could be written in various different notations, such as $(3, 11) + t(7, -9)$ or $\begin{pmatrix} 3 \\ 11 \end{pmatrix} + t\begin{pmatrix} 7 \\ -9 \end{pmatrix}$.

2. Two sides of a triangle in three-dimensional space are formed by the vectors $3\vec{i} + \vec{j} + \vec{k}$ and $\vec{i} + 2\vec{k}$. Find the area of the triangle.

   One method is to take $\frac{1}{2}$ the length of the cross product of the given vectors. Since $(3\vec{i} + \vec{j} + \vec{k}) \times (\vec{i} + 2\vec{k}) = 2\vec{i} - 5\vec{j} - \vec{k}$, the area of the triangle equals $\frac{1}{2}\sqrt{4 + 25 + 1} = \frac{1}{2}\sqrt{30} \approx 2.7386$.

3. How should the number m be chosen to make the two vectors $(1, m)$ and $(2, 3)$ perpendicular?

   The vectors are perpendicular if their dot product is equal to 0: namely, if $2 + 3m = 0$. That means that $m = -2/3$.

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4. Find a reduced matrix equivalent to the matrix $\begin{pmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Dividing each row by a suitable constant reduces the matrix to $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Subtracting the top row from the bottom row reduces the matrix to $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. This matrix is a correct answer, and so is any of the six permutations (rearrangements) of the rows.

The standard "echelon form" of the answer is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

5. Either express the vector $(1, 2, 3)$ as a linear combination of the vectors $(1, 0, 1)$, $(1, 1, 0)$, and $(1, 1, 1)$ or show that it is impossible to do so.

The goal is to solve $x \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ for x, y, and z. Subtracting the second row from the first row gives the equivalent system $x \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$. Subtracting the first row from the third row gives $x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$. Subtracting the third row from the second row gives $x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix}$. So $x = -1$, $y = -2$, $z = 4$, and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Other solution strategies are possible, but the final answer is unique.