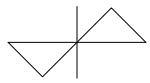
Exam 1 **Math 312** **16 February 1994**

1. This problem discusses different ways to represent the identity function $x \mapsto x$ on the interval $(0, \pi)$ by a series of sines. You may wish to use the integration by parts formula $\displaystyle\int x \sin(kx)\, dx = \frac{-x\cos(kx)}{k} + \frac{\sin(kx)}{k^2}$.

(a) By considering the standard Fourier series for the odd function $f(x) = x$ on the interval $(-\pi, \pi)$, find numbers b_n such that $x = \displaystyle\sum_{n=1}^{\infty} b_n \sin(nx)$ for $-\pi < x < \pi$ (and in particular for $0 < x < \pi$).

(b) Now consider the odd function g on the interval $(-2\pi, 2\pi)$ such that $g(x) = x$ for $0 < x \leq \pi$ and $g(x) = 2\pi - x$ for $\pi \leq x < 2\pi$ (see the diagram). By computing its Fourier series, find numbers B_n such that $x = \displaystyle\sum_{n=1}^{\infty} B_n \sin(nx/2)$ for $-\pi < x < \pi$ (and in particular for $0 < x < \pi$).
Hint: half of these coefficients are zero on symmetry grounds.



(c) Explain why it is *not* possible to find numbers β_n such that an equation of the form $x = \displaystyle\sum_{n=1}^{\infty} \beta_n \sin(2nx)$ holds for $0 < x < \pi$.

2. Consider the heat equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ on the interval $[0, \pi]$, where I have set the diffusivity k equal to 1 for convenience. The boundary conditions are $u(0, t) = 0$ and $\dfrac{\partial u}{\partial x}(\pi, t) = 0$, and the initial condition is $u(x, 0) = x$.

(a) What is the steady-state solution?

(b) Separate variables and solve the resulting pair of one-variable ordinary differential equations by invoking the boundary conditions.

(c) Use superposition and the initial condition to find the final solution to the heat equation.
Hint: Use problem 1 to help identify the needed Fourier coefficients.

(d) Taking the limit as $t \to 0$ in the boundary condition $\dfrac{\partial u}{\partial x}(\pi, t) = 0$ gives $\dfrac{\partial u}{\partial x}(\pi, 0) = 0$. On the other hand, differentiating the initial condition $u(x, 0) = x$ gives $\dfrac{\partial u}{\partial x}(x, 0) = 1$, and taking the limit as $x \to \pi$ gives $\dfrac{\partial u}{\partial x}(\pi, 0) = 1$. Therefore $1 = 0$. What went wrong?