Introduction to Bessel's equation

You computed $\nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\cdot\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\cdot\dfrac{\partial^2}{\partial \theta^2}$ for the Laplace operator in polar coordinates, and you saw that separating variables to solve the homogeneous Laplace equation leads to the equation $r^2R'' + rR' - n^2R = 0$ for the radial part $R(r)$. This singular differential equation is an Euler equidimensional equation that can be solved by trying a solution of the form $R(r) = r^m$ and finding the allowed exponent m.

Solving the heat equation $\nabla^2 u = \dfrac{\partial u}{\partial t}$ or the wave equation $\nabla^2 u = \dfrac{\partial^2 u}{\partial t^2}$ by separation of variables in polar coordinates leads to a different equation due to the separation constant. The simplest case of this equation is *Bessel's equation of order zero*:

$$r^2R'' + rR' + r^2R = 0.$$

Bessel's equation of order μ is $r^2R'' + rR' + (r^2 - \mu^2)R = 0$.

1. Show that Bessel's equation of order zero can also be written

   $$r(rR')' + r^2R = 0.$$

2. Use the chain rule to show that if $f(r)$ is a solution to Bessel's equation of order zero, then $f(\lambda r)$ is a solution to the equation

   $$r^2R'' + rR' + \lambda^2 r^2 R = 0.$$

   (This equation shows up when you separate variables in polar coordinates.)

3. Bessel's equation of any order can be solved by the method of infinite series. Arguing by analogy with the Euler equidimensional equation, it is reasonable to try a series solution of the form $r^m \sum_{n=0}^{\infty} c_n r^n$. One plugs this guess into the equation and matches powers of r to determine first m and then the coefficients c_n recursively.

   Since Bessel's equation has a second-order derivative, there should be two linearly independent solutions. The solution that is typically of physical interest is the one that is bounded at $r = 0$. This solution, appropriately normalized, is usually denoted $J_\mu(r)$.

(a) Carry out the series solution to determine $J_0(r)$ as an infinite series.[1] The standard normalization for the Bessel function of order zero is $c_0 = 1$.

Compare the result to the power series expansion of the cosine function. Based on the power series, can you guess roughly how the graph of J_0 must look?

(b) Use the series method to find $J_{1/2}$, the Bessel function of order $1/2$. This function satisfies the equation

$$r^2R'' + rR' + (r^2 - \tfrac{1}{4})R = 0.$$

The usual normalization for this solution is $c_0 = \sqrt{2/\pi}$.

By examining the power series, can you relate the answer to the sine function?

[1] Answer: $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{r}{2}\right)^{2k}$.