Convergence of Fourier series

You know how to compute the Fourier series of a function f: the coefficients are given by certain integrals. For example, if the function has period 2π, then its Fourier series in complex form looks like $\sum_{n=-\infty}^{\infty} c_n e^{inx}$, where the coefficient c_n is $(2\pi)^{-1}$ times the integral $\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$. (The complex form of the Fourier series is related to the real form via Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.)

Similarly, you can expand a function in a Fourier-Bessel series using Bessel functions, or in a Fourier-Legendre series using Legendre polynomials, or in a Fourier-(`your favorite system of orthogonal functions`) series. In general, if $\{\varphi_n\}$ is a system of (possibly complex-valued) orthogonal functions on an interval $[a, b]$, we associate to f a series $\sum_n c_n\varphi_n(x)$, where

$$c_n = \frac{\int_a^b f(x)\overline{\varphi_n(x)}\,dx}{\int_a^b |\varphi_n(x)|^2\,dx} \tag{1}$$

(the bar denoting complex conjugation).

So far, you have been rather casually writing $f(x) = \sum_n c_n\varphi_n(x)$, but this makes you guilty of aggravated symbolic abuse. You should really be writing some other symbol in place of "="; for example, $f(x) \sim \sum_n c_n\varphi_n(x)$. The trouble is that (a) the series might not converge, and (b) the series might converge, but not to f.

In this course, we have not seen any cases of the pathological situation (a), but there do exist weird functions f that are integrable (so that their Fourier coefficients are defined), yet have Fourier series that diverge everywhere. You are not likely to encounter such functions walking around College Station, however.

The situation (b), on the other hand, is one that we have encountered. Indeed, we had a theorem to the effect that the Fourier series of a piecewise-smooth function does converge at all points, but at jumps of the function f it converges to the midpoint of the jump, which may not be the value of f.

1. Determine the series $\sum_{n=1}^{\infty} c_n \sin(nx)$ associated to the even function $f(x) = x^2$ on $[-\pi, \pi]$ via formula (1) with $\varphi_n(x) = \sin(nx)$.
   Hint: if you are integrating by parts, then you are working too hard.

2. The series $\sum_n c_n \sin(nx)$ converges, but not to x^2. What went wrong?

It is not difficult to demonstrate the convergence of a Fourier series to the associated function if you do not insist on the most general hypotheses. Let's suppose that f (not the same f as above) is a continuous function, periodic of period 2π, and that f is differentiable at 0. (The point 0 is chosen for convenience, but the argument will carry over to a general point.) Fill in the following outline to show that the Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ does converge to $f(x)$ at $x = 0$; in other words, to show that the series $\sum_{n=-\infty}^{\infty} c_n$ converges to $f(0)$.

3. Define a new function $g(x)$ to equal $\dfrac{f(x) - f(0)}{e^{ix} - 1}$ when $0 < x \le \pi$ and $-\pi \le x < 0$, and set $g(0) = -if'(0)$. Show that g is a continuous function on $[-\pi, \pi]$.
4. Let $\sum_{n=-\infty}^{\infty} b_n e^{inx}$ denote the Fourier series of g. Apply Bessel's inequality to deduce that $b_n \to 0$ when $n \to \pm\infty$.
5. Rewrite the definition of g as $g(x)e^{ix} - g(x) = f(x) - f(0)$, multiply both sides by e^{-inx}, and integrate to deduce that $b_{n-1} - b_n = c_n$ if $n \neq 0$, and $b_{-1} - b_0 = c_0 - f(0)$ from the $n = 0$ case.
6. Observe that when you add the coefficients c_n, the previous part gives a telescoping sum. Deduce that $\sum_{n=-\infty}^{\infty} c_n$ converges to $f(0)$ as claimed.

Congratulations! You just proved a convergence theorem for Fourier series.

Fourier, the ninth of the twelve children of his father's second marriage, initially studied for the priesthood. At age twenty-one, he changed plans and became a mathematics teacher. Although imprisoned for a time during the French revolution, he escaped the guillotine and subsequently served Napoleon in various administrative capacities. His most celebrated work is the *Théorie Analytique de la Chaleur* (Analytic Theory of Heat), which applied his method of trigonometric series to the heat equation.

Our Hero
Jean Baptiste Joseph Fourier
1768–1830