More on Bessel functions

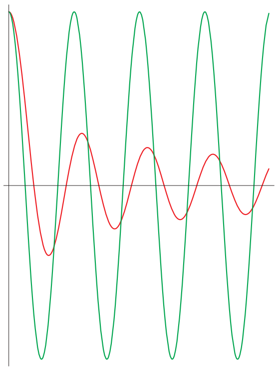
Friedrich Wilhelm Bessel
1784–1846

Bessel, an astronomer, got involved with the functions we call Bessel functions in his study of planetary perturbations. He spent much of his career in determining the positions of some 50,000 stars, and he is noted for discovering the parallax of 61 Cygni.

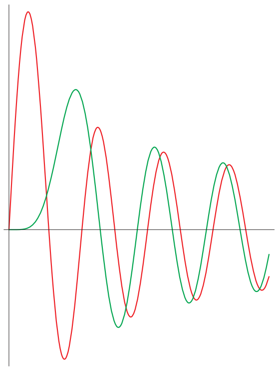
In a sense, Bessel functions should be no more mysterious than trigonometric functions. They are solutions to a linear second-order differential equation, they can be written down as infinite series, they satisfy various identities, and (as we shall see) they have certain orthogonality properties.

Last time you looked at Bessel's differential equation $r^2R'' + rR' + (r^2 - \mu^2)R = 0$. When $\mu = 0$, the solution that is bounded at the origin is Bessel's function of order zero, which has the infinite series expansion $J_0(r) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{r}{2}\right)^{2k}$. If you recall that the infinite series expansion for the cosine function is $\cos r = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} r^{2k}$, then you might guess that the Bessel function J_0 looks something like a damped cosine wave.

It turns out that all the Bessel functions of positive order have graphs that look like lazy trigonometric functions: the graphs oscillate with diminishing amplitude. Unlike the trigonometric functions, however, the Bessel functions generally have zeroes that are not evenly spaced.



J_0 and cosine



J_1 and J_5

The sense in which the Bessel functions resemble trigonometric functions can be made precise. It turns out that when $\mu \geq 0$, there is an approximate formula

$$J_\mu(r) \approx \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\mu\pi}{2} - \frac{\pi}{4}\right) \qquad \text{for large } r \tag{1}$$

in the sense that the two sides have a difference that decays like $r^{-3/2}$. In particular, this means that consecutive zeroes of J_μ are separated by a distance of approximately π.

When $\mu = 1/2$, the approximate formula is actually exact:

$$J_{1/2}(r) = \sqrt{\frac{2}{\pi r}} \sin r. \tag{2}$$

1. Why is equation (2) compatible with equation (1)?

Separating variables in polar coordinates leads to a differential equation of the form

$$r^2 R'' + rR' + \lambda^2 r^2 R = 0, \tag{3}$$

where λ is the eigenvalue. Last time you showed that $J_0(\lambda r)$ is a solution of this equation. The boundary conditions determine the allowed values of the eigenvalue λ.

2. If equation (3) has the boundary condition $R(1) = 0$, then what are the allowed eigenvalues?

3. Equation (3) is a *singular* Sturm-Liouville problem because the coefficient of R'' is zero at $r = 0$. Nonetheless, it is reasonable to expect that the solutions for different eigenvalues might be orthogonal with respect to an appropriate weight function.

   Show that if λ_n and λ_m are different eigenvalues of (3), then the corresponding eigenfunctions are orthogonal on the interval $(0, 1)$ with respect to the weight function r: namely, $\int_0^1 J_0(\lambda_n r) J_0(\lambda_m r)\, r\, dr = 0$.

   One way to show this is to multiply the differential equation for $J_0(\lambda_n r)$ by $J_0(\lambda_m r)$, write the corresponding equation with m and n reversed, subtract, and integrate by parts.