Solving the wave equation on a disk

Problem: solve the wave equation $\nabla^2 u = \dfrac{\partial^2 u}{\partial t^2}$ on the disk centered at the origin with radius 1. The boundary condition is that u equals 0 on the boundary of the disk (which might be a drumhead, for instance). If u is expressed in polar coordinates, then this condition is $u(1, \theta, t) = 0$. The initial conditions are $u(r, \theta, 0) = 1 - r$ and $\dfrac{\partial u}{\partial t}(r, \theta, 0) = 0$.

Carry out the following steps to complete the solution. (See also section 5.7 in the textbook by Powers.)

1. First separation of variables: look for solutions that are products of a spatial function $S(r, \theta)$ and a temporal function $T(t)$. Deduce that $T(t)$ is a trigonometric function, and $\nabla^2 S = -\lambda^2 S$ for some λ to be determined.

2. Second separation of variables: look for S in the form of a product of a radial function $R(r)$ and an angular function $\Theta(\theta)$. Deduce that Θ is a trigonometric function, and R satisfies some form of Bessel's equation.

3. Bring in the boundary condition to deduce that the possible eigenvalues λ are zeroes of Bessel functions of integral order.

4. Write the full solution $u(r, \theta, t)$ as a superposition of product functions. You will have a sum over two summation indices.

5. Bring in the initial conditions to determine the coefficients in the superposition. First use orthogonality for trigonometric functions in the θ variable, and then use orthogonality for Bessel functions in the r variable.

6. Show that the final solution is $u(r, \theta, t) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r) \cos(\lambda_n t)$, where λ_n denotes the nth zero of the Bessel function of order zero, and
$$c_n = \frac{\int_0^1 r(1-r) J_0(\lambda_n r)\, dr}{\int_0^1 (J_0(\lambda_n r))^2\, r\, dr}.$$

7. The solution is independent of the angle θ. How might you have predicted this before working the problem?