The heat equation with transcendental eigenvalues

In the problems you have done so far, solving the heat equation by separation of variables led to an ordinary differential equation $\phi'' + \lambda^2\phi = 0$ that you recognized as having solutions of the form $c_1 \sin(\lambda x) + c_2 \cos(\lambda x)$; and the boundary conditions implied that λ was an integer (or sometimes a half-integer).

More elaborate problems might have (a) a more complicated differential equation, or (b) more complicated boundary conditions. In this exercise, we will examine case (b).

Consider the heat equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ for $0 < x < \pi$ and $0 < t$ with initial condition $u(x,0) = x$ and boundary conditions $u(0,t) = 0$ and $\dfrac{\partial u}{\partial x}(\pi, t) = 100 - u(\pi, t)$. The right-hand boundary condition says that the rate of heat flow across the end of the bar is proportional to the difference between the temperature at the end of the bar and a fixed ambient temperature; this might correspond to heat flow via convection.

1. By finding the steady-state solution and the new boundary conditions for the transient solution and by applying the method of separation of variables, derive the differential equation $\phi'' + \lambda^2\phi = 0$ for the space part of the transient solution.

2. By using the boundary conditions for ϕ at 0 and π, show that the eigenvalue λ must satisfy the equation

$$\tan(\lambda\pi) = -\lambda. \tag{1}$$

   This is a transcendental equation: that is, the solutions cannot be written down in terms of sums and products and powers and roots. However, it is clear from a graph (Figure 1) that there is an infinite sequence of positive solutions λ_1, λ_2, λ_3, ...

3. Why is it valid to restrict attention to $\lambda > 0$?

4. Numerical approximations to the first few eigenvalues are $\lambda_1 \approx 0.788$, $\lambda_2 \approx 1.672$, $\lambda_3 \approx 2.616$, $\lambda_4 \approx 3.587$, $\lambda_5 \approx 4.569$. (I got these numerical values by using Maple's `fsolve` command.) Can you deduce from this data and Figure 1 an approximate value for λ_{100}?

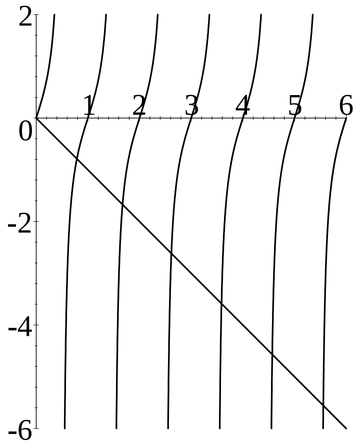

Figure 1: Intersection of $y = \tan(\lambda\pi)$ and $y = -\lambda$

5. Since the eigenvalues λ_n are not integers, you do not know from previous work that the eigenfunctions $\sin(\lambda_n x)$ are orthogonal on the interval $(0, \pi)$. Show that in fact they are orthogonal. (You will need to use equation (1).)

6. Use superposition and orthogonality principles to determine the final solution to the heat equation with the specified side conditions.[1]

[1] Answer: $\dfrac{100x}{1+\pi} + 2(99-\pi)\displaystyle\sum_{n=1}^{\infty} \frac{\cos(\lambda_n \pi)}{\lambda_n(\pi + \cos^2(\lambda_n \pi))} \sin(\lambda_n x) e^{-\lambda_n^2 t}$.