Introduction to Sturm-Liouville problems

Last time, you looked at the differential equation $\phi'' + \lambda^2\phi = 0$ on the interval $(0, \pi)$ with the boundary conditions $\phi(0) = 0$ and $\phi'(\pi) + \phi(\pi) = 0$. You discovered that this boundary value problem does not have a solution that you can write down explicitly. In fact, the eigenfunctions are $\sin(\lambda_n x)$, where the eigenvalues λ_n are solutions of the transcendental equation $\tan(\lambda\pi) = -\lambda$.

Indeed, most differential equations cannot be solved exactly. Therefore it is useful to have some general theorems describing the behavior of typical kinds of differential equations. One class of boundary value problems whose eigenfunctions behave much like sines are the *Sturm-Liouville equations.*[1] These equations have the form

$$(s\phi')' - q\phi + \lambda^2 p\phi = 0,$$

where p and q and s and its derivative s' are assumed to be continuous functions on an interval $[a, b]$. It is also assumed that s and p are positive functions. The middle term is often negative (or missing), so it is conventionally written with a minus sign. The boundary conditions are that some linear combination of $\phi(a)$ and $\phi'(a)$ is zero, and some linear combination of $\phi(b)$ and $\phi'(b)$ is zero.

Many of the standard differential equations of mathematical physics fit the framework of Sturm-Liouville equations. For example, consider the so-called *Hermite equation*:[2]

$$\phi'' - 2x\phi' + \lambda^2\phi = 0.$$

1. At first glance, the Hermite equation does not look like a Sturm-Liouville equation. Show how to recast the Hermite equation in Sturm-Liouville form, and identify the functions s, q, and p.

The most natural domain for the Hermite equation is the whole real line, but for illustrative purposes, let's consider the equation for $0 < x < 1$ with boundary conditions $\phi(0) = 0$ and $\phi(1) - \phi'(1) = 0$. It is not easy to determine the eigenvalues (the values of λ^2 for which this boundary value problem has a non-trivial solution).

[1] Jacques Charles François Sturm lived 1803–1855; born in Switzerland, he spent most of his career in France. The French mathematician Joseph Liouville lived 1809–1882; he is best known for his work in number theory.

[2] The French mathematician Charles Hermite lived 1822–1901. In 1873, he proved that the number e is transcendental.

2. When $\lambda^2 = 0$, the Hermite equation is simple enough to solve almost explicitly. Find the general solution, with two arbitrary constants. (Your solution will contain one unevaluated integral.)

3. Now bring in the boundary conditions. Is $\lambda^2 = 0$ an eigenvalue?

When you studied ordinary differential equations, you saw two methods for solving second-order differential equations with non-constant coefficients (such as the Hermite equation above). One method was to write down an infinite series with undetermined coefficients, plug it into the equation, and find a recurrence formula for the unknown coefficients. Another method was to guess one solution, and apply the technique of reduction of order to find a second solution: that is, try as a second solution an unknown function times the first solution.

4. Use one of these methods to find two linearly independent solutions of the Hermite equation when $\lambda^2 = 2$.
   (One solution is a polynomial, but the other solution can only be written as an infinite series or as an integral.)

5. Now bring in the boundary conditions. Is $\lambda^2 = 2$ an eigenvalue?

At this point you should be convinced that it would be nice to have a theorem about the existence of eigenvalues of Sturm-Liouville problems. The theorem says that there is an infinite sequence of eigenvalues tending to infinity. Associated to each eigenvalue is one eigenfunction, and the nth eigenfunction has $n-1$ zeroes in the open interval (a, b). (Thus the eigenfunctions have oscillatory behavior, much like sines.) The eigenfunctions are orthogonal, but with respect to the *weight function* p: that is, each pair of distinct eigenfunctions ϕ_n and ϕ_m has the property that $\int_a^b \phi_n(x)\phi_m(x)p(x)\,dx = 0$. Finally, if you write the generalized Fourier expansion of a piecewise smooth function in terms of the eigenfunctions of a Sturm-Liouville problem, then the series converges to the function at points of continuity and to the midpoint of the jump at discontinuities. (However, convergence to the function is not guaranteed at the endpoints a and b.)

6. From your work above and the statement of the theorem, you should be able to identify the smallest eigenvalue of the Hermite equation with the specified boundary conditions. What is this value?

7. For the Hermite equation with the specified boundary conditions on the interval $[0, 1]$, what orthogonality relation must the eigenfunctions satisfy?