The heat equation on unbounded domains

Heat flow in a very long bar might be modeled by the heat equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ on a half-line $(0, \infty)$. Some new features arise in this setting.

1. Since the right-hand end of the bar is "at infinity," there is a boundary condition only at the left-hand end. Suppose that the left-hand end of the bar is insulated, which corresponds to the boundary condition $\dfrac{\partial u}{\partial x}(0, t) = 0$. Show that when you use separation of variables, there are solutions corresponding to both positive values and negative values of the separation constant.

(Aside: the non-trigonometric solutions can be written either in terms of exponential functions, or in terms of hyperbolic functions. Recall that these functions are connected via

$$\cosh t = \frac{e^t + e^{-t}}{2}, \qquad \sinh t = \frac{e^t - e^{-t}}{2}$$
$$e^t = \cosh t + \sinh t, \qquad e^{-t} = \cosh t - \sinh t.$$

That is, the hyperbolic functions are just special linear combinations of growing and decaying exponential functions.)

2. The solutions corresponding to positive separation constants and the solutions corresponding to negative separation constants are both valid mathematically, but only one type of solution is sensible physically. Which one, and why?

For the heat equation on an unbounded domain, the eigenvalues form a continuum instead of a sequence. In order to superpose to find the final solution, we will have to work out a Fourier *integral* instead of a Fourier *series*. We will look at Fourier integrals next time.