Some improper integrals

To complete the solution of the heat equation on a half-line, we need to compute Fourier integrals. Here is a warm-up on some improper integrals that arise in this context.

1. (a) Integrate twice by parts to show that

$$\int e^{-px}\sin(x)\,dx = -\frac{e^{-px}\cos(x)}{p^2+1} - \frac{pe^{-px}\sin(x)}{p^2+1}.$$

   (b) Evaluate the double integral $\displaystyle\int_0^\infty\int_0^\infty e^{-px}\sin(x)\,dx\,dp$ by using the above result. Then integrate in the other order[1] and compare the answers to deduce that

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}. \tag{1}$$

   This integral can also be evaluated by the technique of contour integration, which you can learn about in Math 407.

2. (a) Use equation (1) to find (easily!) the value of $\displaystyle\int_0^\infty \frac{\sin(ax)}{x}\,dx$, where a is a constant. (Notice that a might be either positive or negative, and the sign of a will affect your answer.)

   (b) Use the double-angle identity $\sin(2x) = 2(\sin x)(\cos x)$, the previous part, and integration by parts to show that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

3. Evaluate $\displaystyle\int_0^\infty \frac{\sin(ax)}{x}\cos(bx)\,dx$, where a and b are constants. Your answer should depend on whether $|a| < |b|$, $|a| = |b|$, or $|a| > |b|$.
   Hint: no integration is necessary.

[1] If you are worried about justifying the interchange of orders of integration, you can do $\int_a^b\int_c^d$ and then take the limit as a and c tend to 0 and as b and d tend to ∞.