Solving the heat equation on an unbounded domain by using Fourier integrals

Fourier *series* are useful for representing functions on a finite interval (since such functions can be extended to the whole real line as periodic functions). However, to represent a function that is not periodic and that lives on the whole line (or perhaps on a half-line), we need to superpose trigonometric functions with arbitrary periods: this requires a Fourier *integral.*

If f is a piecewise smooth function on the real line such that the improper integral $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges, then the Fourier integral coefficients are defined as follows:

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\lambda x)\,dx, \qquad B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\lambda x)\,dx.$$

The Fourier integral representation of f is then

$$\int_0^{\infty} \left(A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)\right)\,d\lambda.$$

As was the case for Fourier series, this improper integral converges to $f(x)$ at points where f is continuous, and the integral converges to the midpoint of the jump at discontinuities.

When you considered the heat equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ on a half-line $(0,\infty)$ with the boundary condition $\dfrac{\partial u}{\partial x}(0,t) = 0$, you found that functions of the form $\cos(\lambda x)e^{-\lambda^2 t}$ are solutions for every value of λ. Superpositions of such functions are also solutions, so we might expect to match an initial condition by forming a continuous superposition of the form

$$\int_0^{\infty} A(\lambda)\cos(\lambda x)e^{-\lambda^2 t}\,d\lambda.$$

Suppose the initial condition is $u(x,0) = e^{-x}$. Complete the solution[1] of the heat equation by computing an appropriate Fourier integral.
(Since the boundary condition led to a cosine integral, it is appropriate to extend the function $u(x,0)$ to the whole real line as an even function.)

[1] Answer: $u(x,t) = \dfrac{2}{\pi}\displaystyle\int_0^{\infty} \frac{\cos(\lambda x)e^{-\lambda^2 t}}{1+\lambda^2}\,d\lambda.$

Here is a plot of the solution at $t = 0$, $t = 1$, and $t = 2$. Which graph is which?