Solving the wave equation with Fourier series

So far we have studied the heat equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{k} \cdot \dfrac{\partial u}{\partial t}$, where the constant k represents certain physical characteristics of the material in which heat is flowing (the diffusivity). The techniques we used (separation of variables and Fourier analysis) are applicable to other partial differential equations.

One such equation is the *wave equation*, which describes the propagation of sound waves, water waves, electromagnetic waves, and so forth. The wave equation has the following form:[1]

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial^2 u}{\partial t^2},$$

where the constant c represents the propagation speed of the wave. To keep things simple, let's assume for the moment that the physical units have been chosen so that $c = 1$.

The wave equation differs from the heat equation because the wave equation has a second-order derivative with respect to time. Consequently, we should expect to have *two* initial conditions at time $t = 0$: we need to know both $u(x, 0)$ and $\dfrac{\partial u}{\partial t}(x, 0)$ in order to determine a solution.

We also need boundary conditions, which have the same flavor as in the case of the heat equation. Let's suppose we are working on the interval where $0 < x < \pi$, trying to describe the motion of a vibrating string on a musical instrument such as a banjo, cello, charango, dulcimer, guitar, harp, lute, lyre, mandolin, pipa, qin, sitar, ukulele, viola, violin, zither, or балалайка. The most natural boundary conditions would be that the string does not move at its endpoints: in other words, $u(0, t) = 0$ and $u(\pi, t) = 0$.

1. Use separation of variables and Fourier series to solve the wave equation on the interval $(0, \pi)$ with zero boundary conditions and with the initial conditions

$$u(x, 0) = -x^2 + \pi x, \qquad \frac{\partial u}{\partial t}(x, 0) = 0.$$

[1] For a derivation of the wave equation in the setting of a vibrating string, see section 3.1 of our textbook by Powers.

These conditions correspond to the string being pulled into a parabolic shape and then released at time $t = 0$. Your answer will come out as an infinite series.[2]

2. Let $g(x)$ denote the extension of $-x^2 + \pi x$ to the whole real line that is odd and periodic of period 2π. Use trigonometric identities to show that your answer above can be converted to the form

$$u(x,t) = (g(x+t) + g(x-t))/2.$$

(The geometric interpretation is that the solution to the wave equation is a linear combination of two waves, one traveling to the right and one traveling to the left.)

3. Sketch the solution $u(x,t)$ for $0 < x < \pi$ when $t = 0$, $t = \pi/2$, $t = \pi$, $t = 3\pi/2$, and $t = 2\pi$.

4. Whenever you have a convergent Fourier series, there are interesting numerical sums hiding in it for special values of the variables.

   (a) By choosing appropriate values of x and t in your work above, evaluate the following sum.

   $$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^3} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \cdots$$

   (b) By differentiating $u(x,t)$ and choosing appropriate values of x and t, evaluate the following sum.

   $$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \cdots$$

(The answers are rational numbers times powers of π.)

[2] Answer: $u(x,t) = \dfrac{8}{\pi} \displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} (\sin(2k-1)x)(\cos(2k-1)t)$.