Solutions to Exam 1 from 1994

1. (a) If we expand the odd function $f(x) = x$ on the interval $(-\pi, \pi)$ in a sine series $\sum_{n=1}^{\infty} b_n \sin(nx)$, the coefficient b_n will equal

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx &= \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx \\ &= \frac{2}{\pi}\left[\frac{-x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[\frac{-\pi\cos(n\pi)}{n} - 0 + 0 - 0\right] = \frac{-2\cdot(-1)^n}{n}.\end{aligned}$$

   (b) If we expand the odd function $g(x)$ on the interval $(-2\pi, 2\pi)$ in a sine series $\sum_{n=1}^{\infty} B_n \sin(nx/2)$, the coefficient B_n will equal

$$\frac{1}{2\pi}\int_{-2\pi}^{2\pi} g(x)\sin(nx/2)\,dx = \frac{1}{\pi}\int_0^{2\pi} g(x)\sin(nx/2)\,dx.$$

   Observe that the function g is symmetric about the midpoint of the interval $(0, 2\pi)$, while the function $\sin(nx/2)$ is anti-symmetric about the midpoint when n is even. Consequently, the above integral is zero when n is even. On the other hand, $\sin(nx/2)$ is symmetric about the midpoint of the interval $(0, 2\pi)$ when n is odd, so in this case we can integrate over the interval $(0, \pi)$ and double the result. Consequently, we find that

$$\begin{aligned}B_{2k+1} &= \frac{2}{\pi}\int_0^{\pi} x\sin((2k+1)x/2)\,dx \\ &= \frac{2}{\pi}\left[\frac{-x\cos((2k+1)x/2)}{(2k+1)/2} + \frac{\sin((2k+1)x/2)}{((2k+1)/2)^2}\right]_0^{\pi} \\ &= \frac{2}{\pi}\left[0 - 0 + \frac{\sin((2k+1)\pi/2)}{((2k+1)/2)^2} - 0\right] = \frac{8\cdot(-1)^k}{\pi(2k+1)^2}.\end{aligned}$$

   (c) The most brutal reason that it is not possible to find coefficients β_n such that $x = \sum_{n=1}^{\infty} \beta_n \sin(2nx)$ on the interval $(0, \pi)$ is that if you plug in $x = \pi/2$, then the right-hand side is 0 and the left-hand side is not 0.

   The underlying explanation is that a function represented by a linear combination of the functions $\sin(2nx)$ is automatically both

odd and periodic of period π. If you start with the function x on the interval $(0, \pi)$, then you can extend it to be either odd, or periodic of period π, but not both simultaneously. In fact, if you extend the function to be periodic of period π and then work out its full Fourier series, you will discover the representation $x = \dfrac{\pi}{2} - \displaystyle\sum_{n=1}^{\infty} \frac{\sin(2nx)}{n}$ when $0 < x < \pi$.

2. (a) The steady-state solution must satisfy the differential equation $\dfrac{\partial^2 u}{\partial x^2} = 0$ with boundary conditions $u(0) = 0$ and $\dfrac{\partial u}{\partial x}(\pi) = 0$. The solution is $u \equiv 0$.

   This makes sense physically. The left-hand end of the bar is sitting on a block of ice, while there is no compensating heat coming through the insulated right-hand end of the bar. After a long time, the whole bar will come to freezing temperature.

   (b) To look for the transient solution (which in this case is the whole solution, since the steady-state solution is zero), we try a solution of the form $u(x,t) = \phi(x) \cdot T(t)$. Plugging into the differential equation gives $\phi''(x) \cdot T(t) = \phi(x) \cdot T'(t)$, or $\phi''(x)/\phi(x) = T'(t)/T(t)$. The left-hand side has zero t-derivative, so the right-hand side must also; and the right-hand side has zero x-derivative, so the left-hand side must also. Consequently, both sides must be constant.

   If this constant is positive, then ϕ will be a linear combination of an increasing exponential and a decaying exponential (or of a sinh and a cosh), and it is easy to see that the boundary conditions cannot be matched by such a linear combination. The constant cannot be zero, because then ϕ would be a first-degree polynomial, and we saw in part 2a that the only such solution is identically zero. Therefore the constant must be negative; call it $-\lambda^2$.

   Then $T(t) = e^{-\lambda^2 t}$ and $\phi(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$. Applying the boundary condition at $x = 0$ gives $0 = \phi(0) = c_1$. Applying the boundary condition at $x = \pi$ then gives $0 = \phi'(\pi) = c_2 \lambda \cos(\lambda \pi)$. Therefore (if we are to avoid the trivial solution), the value of λ must be half an odd integer: $\lambda = (2k+1)/2$ for some integer k. Since the sine function is anti-symmetric, we may as

well assume that $k \geq 0$.

In summary, a function of the form $e^{-(2k+1)^2 t/4}\sin((2k+1)x/2)$ satisfies the differential equation and the boundary conditions.

(c) Since the boundary conditions are homogeneous, we can superpose the solutions just found. Therefore a series of the form $\sum_{k=0}^{\infty} B_{2k+1} e^{-(2k+1)^2 t/4}\sin((2k+1)x/2)$ will satisfy the differential equation and the boundary conditions. To determine the coefficients B_{2k+1}, we bring in the initial condition. Setting $t = 0$ gives that $\sum_{k=0}^{\infty} B_{2k+1}\sin((2k+1)x/2) = x$ when $0 < x < \pi$.

This is exactly the situation of problem 1b above, so we already know the value of B_{2k+1}. Consequently, the final solution to the heat equation is

$$u(x,t) = \sum_{k=0}^{\infty} \frac{8\cdot(-1)^k}{\pi(2k+1)^2} e^{-(2k+1)^2 t/4}\sin((2k+1)x/2).$$

(d) What is wrong is the assumption that the derivative $\partial u/\partial x$ is necessarily continuous at the point where $t = 0$ and $x = \pi$. Indeed, the initial condition says that heat is flowing across the right-hand end of the bar just before $t = 0$. At time 0, we suddenly clamp an insulator on the right-end hand of the bar and cut off the heat flow. That is a discontinuous change in the physical conditions, and it is reflected by a corresponding discontinuity in the mathematical solution.