Introduction to the heat equation

The problem is to describe the temperature in a bar of iron. To keep things simple, suppose that heat flows into or out of the bar only at its ends. Also suppose that the bar is long and thin, so that the temperature depends only on one spatial dimension x.

The *heat equation* says that the temperature $u(x,t)$ in the bar at position x and time t satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}.$$

(In general, there will be a constant factor in the equation that depends on the physical characteristics of the material of the bar, but for the moment take the constant to be 1.)

As you know from studying ordinary differential equations, there need to be some side conditions to fix the solution uniquely. (For example, if you know the velocity of a moving automobile, you will not know its position unless you know where it started.)

Suppose that the left-hand end of the bar, at position $x = 0$, is held at temperature -10 for all positive time, and the right-hand end of the bar, at position $x = 1$, is held at temperature 60 (these are *boundary conditions*). Suppose that at time $t = 0$, the temperature in the bar at position x is x^2 (this is an *initial condition*).

1. Draw a diagram in $x\,t$ space and label it appropriately with the indicated side conditions for the temperature u.

2. One expects the solution $u(x,t)$ to be the sum of a *steady state solution* $u_{\text{steady state}}(x)$ that is independent of the time variable and a *transient solution* $u_{\text{transient}}(x,t)$ that tends to 0 when $t \to \infty$. Find the differential equation and side conditions for the steady state solution.

3. Solve the differential equation to find the steady state solution.

4. Suppose the initial condition is changed. What effect does this have on the steady state solution?

5. Find the differential equation and side conditions for the transient solution.

6. Suppose the boundary conditions are changed. What effect does this have on the transient solution? What superposition principle does the transient solution satisfy?

Coming attraction: how to calculate the transient solution.