Exercise on separation of variables

As in the previous handout, consider the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \cdot \frac{\partial u}{\partial t}$$

with initial condition $u(x,0) = x^2$.
Let's look at five different sets of boundary conditions:

1. $u(0,t) = -10$ and $u(1,t) = 60$
2. $u(0,t) = -10$ and $\dfrac{\partial u}{\partial x}(1,t) = 0$
3. $\dfrac{\partial u}{\partial x}(0,t) = 0$ and $\dfrac{\partial u}{\partial x}(1,t) = 0$
4. $u(0,t) = -10$ and $\dfrac{\partial u}{\partial x}(1,t) = \dfrac{u(1,t) - 50}{2}$
5. $\dfrac{\partial u}{\partial x}(0,t) = 0$ and $\dfrac{\partial u}{\partial x}(1,t) = \dfrac{u(1,t) - 50}{2}$

For each set of boundary conditions, answer the following questions:

(a) What is the physical interpretation of the boundary conditions?

(b) Find the steady-state solution.

(c) Find the new boundary conditions and new initial condition that the transient solution satisfies.

(d) Find all product functions of the form $\phi(x) \cdot T(t)$ that satisfy the heat equation with the new (homogeneous) boundary conditions. You should expect to find a family of such functions indexed by an integer n. (In fancy language, you are solving an *eigenvalue problem.*)

(e) If the transient solution is a superposition $\sum_n c_n \phi_n(x) T_n(t)$, then what equation must $\sum_n c_n \phi_n(x)$ satisfy?

Coming attraction: how to find the coefficients c_n.