Care and feeding of trigonometric identities

Do you remember all those addition formulas and double-angle identities for trigonometric functions? Of course, you could look them up,[1] but better is to remember how to derive the identities. It is easy if you know complex numbers and the famous formula of Euler[2]

$$e^{ix} = (\cos x) + i(\sin x)$$

that relates the complex exponential function to the real trigonometric functions.

Here is an example of how to derive the double-angle formulas.

$$\begin{aligned} e^{2ix} &= \left(e^{ix}\right)^2, \text{ so by Euler's formula} \\ \cos(2x) + i\sin(2x) &= (\cos x + i \sin x)^2 \\ &= \left(\cos^2 x - \sin^2 x\right) + 2i(\cos x)(\sin x). \end{aligned}$$

Matching the real and imaginary parts gives $\cos(2x) = \cos^2 x - \sin^2 x$ and $\sin(2x) = 2(\cos x)(\sin x)$.

1. Derive the addition formulas for $\cos(x + y)$ and $\sin(x + y)$ from
$$e^{i(x+y)} = e^{ix} \cdot e^{iy}.$$

2. By adding or subtracting the formulas for $\cos(x \pm y)$ and $\sin(x \pm y)$, derive the three formulas for expressing products of sines and cosines in terms of sums (e.g., $(\sin x)(\sin y) = \frac{1}{2}(\cos(x - y) - \cos(x + y))$).

3. Use the above formulas to prove the following *orthogonality relations* for trigonometric functions (where n and m denote integers).
$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0,$$
$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = 0 = \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx, \text{ if } n \neq m,$$
$$\int_{-\pi}^{\pi} \sin(nx)\sin(nx)\,dx = \pi = \int_{-\pi}^{\pi} \cos(nx)\cos(nx)\,dx.$$

[1] See, for instance, the appendix in our textbook by Powers, page 359.
[2] Pronounced "Oiler."