Solving the heat equation with Fourier series

We have learned that the method of separation of variables reduces the problem of solving the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \cdot \frac{\partial u}{\partial t}$$

to a problem in Fourier series; and we have learned how to compute Fourier coefficients. In this exercise, you will put it all together to find the complete solution of the heat equation with certain specified side conditions.

One point to keep in mind: we saw last time that the Fourier series you get for a function defined on a half-interval depends on what extension of the function you use (symmetric, anti-symmetric, or neither). In solving the heat equation, you will need to choose the extension to be compatible with the kind of series (sine series or cosine series) that you need to match the boundary conditions.

1. Solve the heat equation on the interval $[0, \pi]$ if the initial temperature is $u(x, 0) = x$, and starting at time $t = 0$ the left-hand end of the bar (at position $x = 0$) is placed in ice water, and the right-hand end of the bar (at position $x = \pi$) is placed in boiling water.

   After you have solved the problem, you may check your result against my answer,[1] but don't peek until you have worked the problem yourself!

2. Solve the heat equation on the interval $[0, \pi]$ if the initial temperature is $u(x, 0) = x$, and starting at time $t = 0$ both ends of the bar are wrapped in insulating material.

3. For both problems, check to see that your answers make sense. Do the solutions behave properly when $t \to \infty$? when $x = 0$? when $x = \pi$?

[1] The answer is $u(x,t) = \frac{100x}{\pi} + \left(\frac{200}{\pi} - 2\right) \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (\sin nx) e^{-kn^2 t}$ if the temperature is measured in °C.