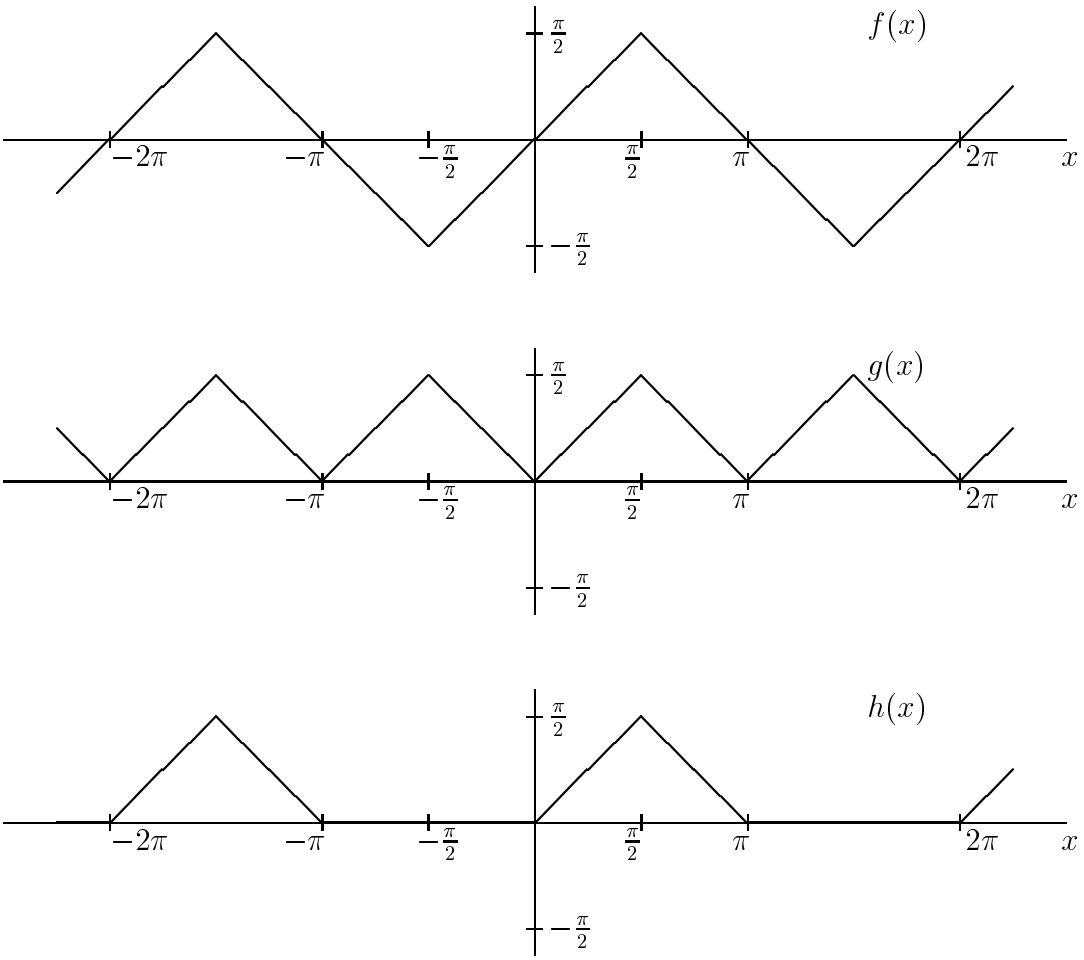
Some Fourier series solutions

Last time you worked on computing the Fourier series $\sum_{n=0}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$ for the three functions f, g, and h shown below. Here are solutions.



Since f is an odd (anti-symmetric) function, its Fourier series will contain only sine terms. That is, the coefficients a_n, which arise by integrating f times cosine functions, will be zero for symmetry reasons.

To find the coefficients b_n, we need to compute the following integral:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,dx.$$

(Because the product of two odd functions is an even function, the computation can be simplified by integrating over half the interval and doubling the result.)

Now since $f(x) = x$ when $0 < x < \pi/2$, and $f(x) = \pi - x$ when $\pi/2 < x < \pi$, we have

$$b_n = \frac{2}{\pi}\left(\int_0^{\pi/2} x\sin(nx)\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin(nx)\,dx\right).$$

Evaluating the first integral via integration by parts gives

$$\begin{aligned}\int_0^{\pi/2} x\sin(nx)\,dx &= x\cdot\frac{(-\cos nx)}{n}\bigg|_0^{\pi/2} - \int_0^{\pi/2}\frac{(-\cos nx)}{n}\,dx \\ &= x\cdot\frac{(-\cos nx)}{n}\bigg|_0^{\pi/2} + \frac{\sin(nx)}{n^2}\bigg|_0^{\pi/2} \\ &= \frac{\pi}{2}\cdot\frac{(-\cos(n\pi/2))}{n} - 0 + \frac{\sin(n\pi/2)}{n^2} - 0.\end{aligned}$$

The second integral can be computed similarly:

$$\begin{aligned}\int_{\pi/2}^{\pi}(\pi - x)\sin(nx)\,dx &= (\pi - x)\cdot\frac{(-\cos nx)}{n}\bigg|_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\frac{(-\cos nx)}{n}(-dx) \\ &= (\pi - x)\cdot\frac{(-\cos nx)}{n}\bigg|_{\pi/2}^{\pi} - \frac{\sin(nx)}{n^2}\bigg|_{\pi/2}^{\pi} \\ &= 0 - \frac{\pi}{2}\cdot\frac{(-\cos(n\pi/2))}{n} - 0 + \frac{\sin(n\pi/2)}{n^2}.\end{aligned}$$

Adding the two integrals together yields $b_n = \dfrac{4\sin(n\pi/2)}{\pi n^2}$, and therefore the Fourier series for f is $\displaystyle\sum_{n=1}^{\infty}\frac{4\sin(n\pi/2)}{\pi n^2}\cdot\sin(nx)$.

You can rewrite this series in a simpler form if you observe that $\sin(n\pi/2)$ is 0 when n is an even integer. Moreover, when n is an odd integer the factor $\sin(n\pi/2)$ alternates between $+1$ and -1. Therefore, another expression for the Fourier series of f is $\dfrac{4}{\pi}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin((2n+1)x)$.

The function g is even, so its Fourier series will have no sine terms: $b_n = 0$. To find a_n when $n \neq 0$, we compute as follows:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} g(x)\cos(nx)\,dx \\ &= \frac{2}{\pi}\left(\int_0^{\pi/2} x\cos(nx)\,dx + \int_{\pi/2}^{\pi}(\pi - x)\cos(nx)\,dx\right). \end{aligned}$$

Integrating by parts gives

$$\begin{aligned} \int_0^{\pi/2} x\cos(nx)\,dx &= \left[\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_0^{\pi/2} \\ &= \frac{\pi}{2}\cdot\frac{\sin(n\pi/2)}{n} - 0 + \frac{\cos(n\pi/2)}{n^2} - \frac{1}{n^2} \end{aligned}$$

for the first integral and

$$\begin{aligned} \int_{\pi/2}^{\pi}(\pi - x)\cos(nx)\,dx &= \left[\frac{(\pi - x)\sin(nx)}{n} - \frac{\cos(nx)}{n^2}\right]_{\pi/2}^{\pi} \\ &= 0 - \frac{\pi}{2}\cdot\frac{\sin(n\pi/2)}{n} - \frac{\cos(n\pi)}{n^2} + \frac{\cos(n\pi/2)}{n^2} \end{aligned}$$

for the second integral.

Adding the results shows that $a_n = \dfrac{4\cos(n\pi/2)}{\pi n^2} - \dfrac{2(1+\cos(n\pi))}{\pi n^2}$. If n is an odd integer, then $\cos(n\pi/2) = 0$ and $\cos(n\pi) = -1$, so $a_n = 0$. If n is even, then n is either a multiple of 4 or else 2 more than a multiple of 4. In the first case, all the cosine terms equal 1, and everything cancels out to give $a_n = 0$ again. In the second case, a_n reduces to $-8/(\pi n^2)$.

Since we cannot divide by 0, we must find the constant term in the Fourier series (corresponding to $n = 0$) by a separate calculation. If you try to remember a formula for the constant term, you may get mixed up about whether it is a_0 or $a_0/2$, so it is safer to remember that the constant term is just the average value of the function over a period. Since the region under the graph of g consists of triangles, it is easy to find this average value without integrating: $2\cdot\left(\frac{1}{2}\pi\right)\cdot\frac{\pi}{2}\cdot\frac{1}{2\pi} = \pi/4$. (There are two triangles, the area of each is half the base times the altitude, and you divide by the length of the interval to get the average value.)

Thus, the Fourier series for g is $\dfrac{\pi}{4} - \dfrac{8}{\pi}\displaystyle\sum_{n=0}^{\infty}\frac{1}{(4n+2)^2}\cos((4n+2)x)$.

Since the function h is the average of the functions f and g, you can get its Fourier series by adding the series for f and g together and dividing by 2.