The Laplacian in polar coordinates

In problems involving circular symmetry, it may be convenient to work in polar coordinates. Recall that the radial distance r and the angle θ are related to the rectangular coordinates x and y via $x = r\cos\theta$ and $y = r\sin\theta$.

Derivatives can be converted between different coordinate systems by using the chain rule. For example, $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial \theta}\cdot\dfrac{\partial \theta}{\partial x}$. However, it is rather nastier to compute such derivatives as $\dfrac{\partial r}{\partial x}$ and $\dfrac{\partial \theta}{\partial x}$ than it is to compute $\dfrac{\partial x}{\partial r}$ and $\dfrac{\partial x}{\partial \theta}$.

1. By using the chain rule and computing easy partial derivatives, show that

$$\begin{pmatrix} \partial u/\partial r \\ \partial u/\partial \theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \partial u/\partial x \\ \partial u/\partial y \end{pmatrix}$$

   (matrix multiplication).

2. Invert the matrix to show that

$$\begin{pmatrix} \partial u/\partial x \\ \partial u/\partial y \end{pmatrix} = \begin{pmatrix} \cos\theta & -r^{-1}\sin\theta \\ \sin\theta & r^{-1}\cos\theta \end{pmatrix} \begin{pmatrix} \partial u/\partial r \\ \partial u/\partial \theta \end{pmatrix}.$$

3. In differential operator notation, the preceding result says:

$$\frac{\partial}{\partial x} = (\cos\theta)\cdot\frac{\partial}{\partial r} - \frac{(\sin\theta)}{r}\cdot\frac{\partial}{\partial \theta}$$
$$\frac{\partial}{\partial y} = (\sin\theta)\cdot\frac{\partial}{\partial r} + \frac{(\cos\theta)}{r}\cdot\frac{\partial}{\partial \theta}.$$

   Use these formulas to deduce that the Laplace operator $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ can be written in polar coordinates as

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r}.$$