Laplace's equation in a square

Last time you looked at solving Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in the square defined by $0 < x < 1$ and $0 < y < 1$ with boundary conditions

$$\begin{aligned} u(0,y) &= 0 & u(x,0) &= \sin(\pi x) \\ u(1,y) &= \sin(2\pi y) & u(x,1) &= 0. \end{aligned}$$

The solution strategy was to split the problem into two subproblems, each having homogeneous boundary conditions in one of the variables. Namely, set $u = u_1 + u_2$, where u_1 satisfies the boundary conditions

$$\begin{aligned} u_1(0,y) &= 0 & u_1(x,0) &= \sin(\pi x) \\ u_1(1,y) &= 0 & u_1(x,1) &= 0, \end{aligned}$$

and u_2 satisfies the boundary conditions

$$\begin{aligned} u_2(0,y) &= 0 & u_2(x,0) &= 0 \\ u_2(1,y) &= \sin(2\pi y) & u_2(x,1) &= 0. \end{aligned}$$

For each subproblem, you can separate variables and use the homogeneous boundary conditions in one direction to find the eigenvalues. Then you superpose product solutions and use the nonhomogeneous boundary conditions in the other direction to solve for the Fourier series coefficients.

The final answer is

$$u(x,y) = \frac{\sin(\pi x)\left(e^{\pi y} - e^{(2-y)\pi}\right)}{1 - e^{2\pi}} + \frac{\sin(2\pi y)\left(e^{2\pi x} - e^{-2\pi x}\right)}{e^{2\pi} - e^{-2\pi}}.$$

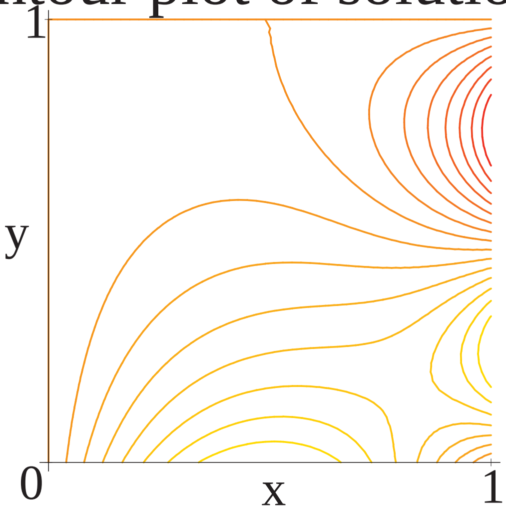
You should be able to verify easily that this solution does satisfy the differential equation and the four boundary conditions. If you recall that the hyperbolic sine function is defined by $\sinh(A) = \frac{1}{2}(e^A - e^{-A})$, then you can massage the answer into the following alternate form:

$$u(x,y) = (\sin(\pi x)) \cdot \frac{\sinh(\pi(1-y))}{\sinh(\pi)} + (\sin(2\pi y)) \cdot \frac{\sinh(2\pi x)}{\sinh(2\pi)}.$$

The second form may show more clearly what is happening on the edges of the square.

Overleaf are a two-dimensional contour plot showing equipotential curves of the solution and a three-dimensional plot of the solution itself.

contour plot of solution



3d plot of solution