The wave equation: D'Alembert's method

Last time you solved the wave equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2} \cdot \dfrac{\partial^2 u}{\partial t^2}$ by the method of separation of variables. After the fact, you determined that the solution was a linear combination of two waves, one traveling to the right and one traveling to the left. It turns out that this is the general situation for the wave equation.

In fact, the wave equation is one of the few partial differential equations that can be solved in closed form. A method for finding the closed-form solution is named after the French mathematician Jean Le Rond d'Alembert.[1]

1. Use the chain rule to show that $u(x,t) = v(x \pm ct)$ satisfies the wave equation, where v is an arbitrary twice-differentiable function.

2. Interpret $v(x+ct)$ and $v(x-ct)$ as traveling waves. Which one moves to the right as t increases?

We may reasonably look for a solution of the wave equation in the form $u(x,t) = \psi(x+ct) + \phi(x-ct)$, where ψ and ϕ are to be determined. We must use the boundary conditions and initial conditions to find the unknown functions ψ and ϕ. For example, let's consider the following side conditions for the wave equation on the interval $(0,\pi)$.

$$u(0,t) = 0 \qquad u(x,0) = \sin(2x)$$
$$u(\pi,t) = 0 \qquad \frac{\partial u}{\partial t}(x,0) = \sin(x)$$

In the method of separation of variables, we use the boundary conditions first, and the initial conditions second; but in d'Alembert's method, we use the side conditions in the reverse order.

3. Use the initial conditions to show that a solution for ψ and ϕ for x between 0 and π is given by

$$\psi(x) = \frac{1}{2}\left(\sin(2x) - \frac{\cos(x)}{c}\right)$$
$$\phi(x) = \frac{1}{2}\left(\sin(2x) + \frac{\cos(x)}{c}\right).$$

[1] Lived 1717–1783. His mother abandoned the infant boy on the steps of the church St. Jean Baptiste le Rond, whence his name.

4. Use the boundary conditions to determine how ψ and ϕ should be defined for $x > \pi$ and for $x < 0$.

5. Show that your solution to the wave equation can be written in the following way.

$$u(x,t) = (\sin(2x))(\cos(2ct)) + \frac{(\sin(x))(\sin(ct))}{c}$$

6. Describe the behavior of the solution when c is very large; when c is very small.