Solutions to Exam 2 from 1994

1. (a) From the initial conditions, we have

$$\psi(x) + \phi(x) = u(x,0) = \sin(x)$$
$$\psi'(x) - \phi'(x) = \frac{\partial u}{\partial t}(x,0) = 0.$$

Integrating the second condition shows that $\psi(x) - \phi(x) = A$, where A is some constant. Combining this equation with the first initial condition (by addition and subtraction) shows that

$$\psi(x) = \frac{1}{2}(\sin(x) + A)$$
$$\phi(x) = \frac{1}{2}(\sin(x) - A)$$

when $0 < x < 2\pi$.

Next we need to see how to extend the definitions of ψ and ϕ to the whole real line so as to satisfy the boundary conditions: namely,

$$\psi(t) + \phi(-t) = u(0,t) = 0$$
$$\psi(2\pi + t) + \phi(2\pi - t) = u(2\pi, t) = 0.$$

Since the sine function is odd and periodic with period 2π, inspection shows that we can use the natural extension of $\sin(x)$ to the whole real line.

Thus the final solution is $u(x,t) = \frac{1}{2}(\sin(x+t) + \sin(x-t))$.

(b) We get $X(0)T(t) = 0$ and $X(2\pi)T(t) = 0$ from the boundary conditions, so if we are to avoid the trivial solution, we must have $X(0) = X(2\pi) = 0$. The solution to the equation $X'' + \lambda^2 X = 0$ is $X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$. The condition $X(0) = 0$ implies that $c_1 = 0$, and the condition $X(2\pi) = 0$ implies that λ is either an integer or a half-integer. Consequently, our solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} (\sin(nx/2))(a_n \cos(nt/2) + b_n \sin(nt/2)).$$

Now $\sin(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx/2)$ by the first initial condition. Using orthogonality, we deduce that $a_n = 0$ if $n \neq 2$, and $a_2 = 1$. The second initial condition implies that $0 = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \frac{n}{2} \cdot b_n \sin(nx/2)$. Orthogonality implies that $b_n = 0$ for every n.

Consequently, the infinite series solution reduces to a single term: $u(x,t) = (\sin(x))(\cos(t))$.

(c) By a standard trigonometric addition formula, we can write

$$(\sin(x))(\cos(t)) = \frac{1}{2}(\sin(x+t) + \sin(x-t)),$$

so the answers to 1a and 1b agree.

(d) $u(x, \pi/2)$ is identically zero, and $u(x,\pi) = -\sin(x)$.

2. This problem is similar to one you have done in class and for homework. Trying a product solution $\phi(x)T(t)$ leads to $\phi''(x) = \pm\lambda^2\phi(x)$ and $T'(t) = \pm\lambda^2 T(t)$. For a physically realistic solution, the temperature must remain bounded, so we discard the plus sign. Then ϕ is a linear combination of sines and cosines. Since the boundary condition implies that $\phi(0) = 0$, we can keep only the sines. Thus the solution is a continuous superposition of the following form:

$$u(x,t) = \int_0^\infty b(\lambda) \sin(\lambda x) e^{-\lambda^2 t}\, d\lambda.$$

The initial condition says $e^{-x} = u(x,0) = \int_0^\infty b(\lambda)\sin(\lambda x)\, d\lambda$, so the Fourier integral coefficient $b(\lambda)$ is given by

$$b(\lambda) = \frac{2}{\pi}\int_0^\infty e^{-x}\sin(\lambda x)\, dx = \frac{2\lambda}{\pi(1+\lambda^2)},$$

and the final solution is

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\lambda \sin(\lambda x) e^{-\lambda^2 t}}{1+\lambda^2}\, d\lambda.$$