Exam 2 **Math 312** **25 March 1994**

1. Consider the wave equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$ on the interval $0 < x < 2\pi$ with the boundary conditions $\left\{ \begin{array}{r} u(0,t) = 0 \\ u(2\pi,t) = 0 \end{array} \right\}$ and the initial conditions $\left\{ \begin{array}{r} u(x,0) = \sin x \\ \dfrac{\partial u}{\partial t}(x,0) = 0 \end{array} \right\}$. (The speed of propagation $c = 1$.)

(a) According to D'Alembert's method, the solution can be written in the form $u(x,t) = \psi(x+t) + \phi(x-t)$ for certain functions ψ and ϕ. Use the initial conditions and the boundary conditions to determine ψ and ϕ on the whole real line, and thereby find the solution $u(x,t)$.

(b) According to the method of separation of variables, the solution is a superposition of product functions $X(x)T(t)$, where $X'' + \lambda^2 X = 0$ and $T'' + \lambda^2 T = 0$ for an eigenvalue λ. Finish the separation of variables solution by using Fourier series to find a series representation for $u(x,t)$.

(c) Verify that your solutions in (a) and (b) are compatible.

(d) Sketch the solution $u(x,t)$ for $t = \pi/2$ and for $t = \pi$.

2. Consider the heat equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ on the unbounded interval $x > 0$ with boundary condition $u(0,t) = 0$ and initial condition $u(x,0) = e^{-x}$. (The diffusivity $k = 1$.) Use the method of separation of variables to find a Fourier integral representation for the solution $u(x,t)$.

You may wish to use the following integration formula number 4.13 from the table in the appendix of the textbook:

$$\int e^{kx} \sin(\lambda x)\, dx = \frac{e^{kx}\big(k \sin(\lambda x) - \lambda \cos(\lambda x)\big)}{k^2 + \lambda^2}.$$