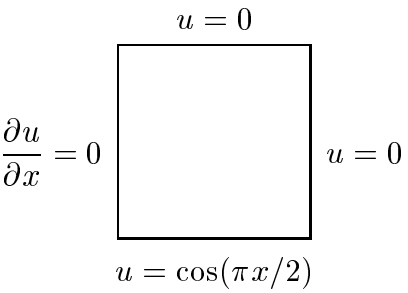
Exam 3 **Math 312** **20 April 1994**

Part I. Solve the following boundary value problem for the potential equation (Laplace's equation) on a square:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{for } 0 < x < 1 \text{ and } 0 < y < 1,$$

with the boundary conditions (as indicated in the diagram)

$$\frac{\partial u}{\partial x}(0, y) = 0,$$
$$u(1, y) = 0,$$
$$u(x, 0) = \cos(\pi x/2),$$
$$u(x, 1) = 0.$$



Part II. Do **one** of the following two problems.

A. Prove the orthogonality relation for the Legendre polynomials:

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0$$

when m and n are different non-negative integers.
Suggestion: Multiply the differential equation defining P_m by P_n and integrate by parts; repeat with m and n reversed and compare.

B. To solve the wave equation on the ball, we separated variables in spherical coordinates. The resulting equation in the radial variable was

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} + (\lambda^2 r^2 - \mu^2) R = 0. \qquad (*)$$

The angular part of the problem implied that the eigenvalue $\mu = n(n+1)$ for a non-negative integer n. A bounded solution to the radial equation $(*)$ is then $R(r) = r^{-\frac{1}{2}} J_{n+\frac{1}{2}}(\lambda r)$, where $J_{n+\frac{1}{2}}$ is the Bessel function of the first kind of order $n + \frac{1}{2}$. For the special case $\lambda = 1$ and $n = 0$, verify that this is a solution to $(*)$.