Solutions to Exam 3 from 1994

Part I To separate variables, we look for a harmonic function $u(x, y)$ in the form of a (not identically zero!) product $X(x)Y(y)$. Then we will try to match the boundary conditions by superposing such functions.

By substituting into Laplace's equation, we obtain $\dfrac{X''}{X} = -\dfrac{Y''}{Y}$. Since the left-hand side is independent of y, and the right-hand side is independent of x, both sides must be constant, say equal to m.

We begin with the x equation $X'' = mX$ because the boundary conditions in the x direction are homogeneous, and hence are preserved under superposition. The two boundary conditions $\dfrac{\partial u}{\partial x}(0, y) = 0$ and $u(1, y) = 0$, applied to our product $u(x, y) = X(x)Y(y)$, yield $X'(0) = 0$ and $X(1) = 0$. There are three cases, depending on m.

Case 1: $m > 0$, say $m = \lambda^2$. Then $X(x) = A\cosh(\lambda x) + B\sinh(\lambda x)$. The first boundary condition implies that $B\lambda\cosh(0) = 0$, whence $B = 0$; and then the second boundary condition implies that $A\cosh(\lambda) = 0$, whence $A = 0$. So this case yields only the trivial solution.

Case 2: $m = 0$. Then $X(x) = Ax + B$, and the boundary conditions imply that $A = 0$ and then $B = 0$. So again there is no non-trivial solution.

Case 3: $m < 0$, say $m = -\lambda^2$. Then $X(x) = A\cos(\lambda x) + B\sin(\lambda x)$. The first boundary condition implies that $B\lambda\cos(0) = 0$, whence $B = 0$; and then the second boundary condition implies that $A\cos(\lambda) = 0$, whence λ must have the form $(2n + 1)\pi/2 := \lambda_n$ for some non-negative integer n.

Having used the homogeneous boundary conditions in the x variable to determine the eigenvalues λ_n, we turn to the y equation $Y'' = -mY = \lambda_n^2 Y$. This has the solution $Y(y) = a_n\cosh(\lambda_n y) + b_n\sinh(\lambda_n y)$. (Alternatively, we could write $Y(y) = c_n e^{\lambda_n y} + d_n e^{-\lambda_n y}$ with the same ultimate result.)

Superposing the product solutions, we can now state that

$$u(x, y) = \sum_{n=0}^{\infty} \big(\cos(\lambda_n x)\big)\big(a_n\cosh(\lambda_n y) + b_n\sinh(\lambda_n y)\big)$$

is a harmonic function that satisfies the first pair of boundary conditions. We need to determine the coefficients a_n and b_n so that this series will also satisfy the second pair of boundary conditions.

From the boundary condition $u(x, 0) = \cos(\pi x/2)$, we find that

$$\cos(\pi x/2) = \sum_{n=0}^{\infty} a_n\cos((2n + 1)\pi x/2).$$

By the uniqueness of Fourier series coefficients, we deduce (since the left-hand side is a Fourier series with only one term) that $a_n = 0$ if $n \neq 0$, and $a_0 = 1$.

On the other hand, we find from the boundary condition $u(x, 1) = 0$ that

$$0 = \sum_{n=0}^{\infty} \big(\cos(\lambda_n x)\big) \big(a_n \cosh \lambda_n + b_n \sinh \lambda_n\big).$$

Fourier analysis implies that $0 = a_n \cosh \lambda_n + b_n \sinh \lambda_n$ for each n. When $n \neq 0$, we already know that $a_n = 0$, whence it follows that $b_n = 0$. When $n = 0$, we know $a_0 = 1$, whence it follows that $b_0 = -\coth(\pi/2) = \dfrac{1+e^{\pi}}{1-e^{\pi}}$.

The final solution, therefore, is

$$u(x, y) = (\cos(\pi x/2)) \left(\cosh(\pi y/2) + \frac{1+e^{\pi}}{1-e^{\pi}} \sinh(\pi y/2) \right).$$

An alternate way to write the solution is

$$u(x, y) = \frac{\cos(\pi x/2)}{1-e^{\pi}} \left(e^{\pi y/2} - e^{\pi} e^{-\pi y/2} \right).$$

Part II, A. First solution The Legendre polynomial P_m satisfies Legendre's equation of order m, that is, $((1-x^2)P_m')' + m(m+1)P_m = 0$. Multiplying by P_n and integrating gives

$$0 = \int_{-1}^{1} P_n(x)\big((1-x^2)P_m'(x)\big)'\,dx + m(m+1)\int_{-1}^{1} P_n(x)P_m(x)\,dx.$$

Integrating the first term by parts yields

$$0 = -\int_{-1}^{1} P_n'(x)(1-x^2)P_m'(x)\,dx + m(m+1)\int_{-1}^{1} P_n(x)P_m(x)\,dx,$$

since the boundary term $P_n(x)(1-x^2)P_m'(x)\big|_{-1}^{1}$ evaluates to zero due to the factor of $(1-x^2)$.

The argument is symmetric in m and n, so interchanging the indices gives

$$0 = -\int_{-1}^{1} P_m'(x)(1-x^2)P_n'(x)\,dx + n(n+1)\int_{-1}^{1} P_m(x)P_n(x)\,dx.$$

Subtracting one equation from the other gives

$$0 = \big(m(m+1) - n(n+1)\big) \int_{-1}^{1} P_m(x) P_n(x)\, dx.$$

Now $m(m+1) - n(n+1)$ is zero only if $m = n$ or $m = -n-1$, so if m and n are distinct non-negative integers, we may divide by this quantity to deduce the orthogonality relation $0 = \int_{-1}^{1} P_m(x) P_n(x)\, dx$.

Second solution If n and m are distinct non-negative integers, then one of them is bigger than the other; suppose $n > m$. Apply the Rodrigues formula

$$P_n(x) = \frac{1}{n!\, 2^n} \frac{d^n}{dx^n} (x^2 - 1)^n$$

(formula (7) on page 278 of the textbook) to get

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{1}{n!\, 2^n} \int_{-1}^{1} P_m(x) \frac{d^n}{dx^n} (x^2 - 1)^n\, dx.$$

Now integrate by parts n times to get

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = \frac{(-1)^n}{n!\, 2^n} \int_{-1}^{1} (x^2 - 1)^n \frac{d^n P_m}{dx^n}\, dx.$$

The boundary terms are all zero because each contains at least one undifferentiated factor of $(x^2 - 1)$.

Now $d^n P_m / dx^n = 0$, because $P_m(x)$ is a polynomial of degree m, while $n > m$. This proves the orthogonality relation.

Part II, B. First solution We want to see that $R(r) = r^{-\frac{1}{2}} J_{\frac{1}{2}}(r)$ satisfies the differential equation

$$r^2 R''(r) + 2r R'(r) + r^2 R(r) = 0. \tag{1}$$

To verify this, we substitute the purported solution into the differential equation and simplify.

The first derivative of $r^{-\frac{1}{2}} J_{\frac{1}{2}}(r)$ is $r^{-\frac{1}{2}} J'_{\frac{1}{2}} - \frac{1}{2} r^{-\frac{3}{2}} J_{\frac{1}{2}}$, and the second derivative is $r^{-\frac{1}{2}} J''_{\frac{1}{2}} - r^{-\frac{3}{2}} J'_{\frac{1}{2}} + \frac{3}{4} r^{-\frac{5}{2}} J_{\frac{1}{2}}$. Plugging into the left-hand side of the equation to be verified, we obtain

$$r^{-\frac{1}{2}} \left(r^2 J''_{\frac{1}{2}} + r J'_{\frac{1}{2}} + \left(r^2 - \tfrac{1}{4}\right) J_{\frac{1}{2}} \right).$$

Is this equal to zero? Yes, because Bessel's equation of order one-half is

$$x^2y'' + xy' + \left(x^2 - \left(\tfrac{1}{2}\right)^2\right) y = 0,$$

and $J_{\frac{1}{2}}$ is a solution of this equation.

Second solution The differential equation (1) has a regular singular point at $r = 0$, so we can solve it with a generalized power series of the form $r^\alpha \sum_{n=0}^{\infty} a_n r^n$. Substituting this series into the equation gives

$$\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n r^{n+\alpha} + \sum_{n=0}^{\infty} 2(n+\alpha)a_n r^{n+\alpha} + \sum_{n=2}^{\infty} a_{n-2} r^{n+\alpha} = 0.$$

The coefficient of each power of r must be zero. From the power r^α we obtain $(\alpha)(\alpha-1) + 2(\alpha) = 0$, whence $\alpha = 0$ or $\alpha = -1$. We look for a bounded solution by using the larger root $\alpha = 0$.

The coefficient of r^1 is $2a_1$, so $a_1 = 0$. For $n \geq 2$, the coefficient of r^n is $n(n-1)a_n + 2na_n + a_{n-2}$, so we obtain the recurrence formula $a_n = \dfrac{-a_{n-2}}{n(n+1)}$. Observe that this is exactly the same recurrence relation that arises in solving Bessel's equation of order one-half (see formula (3) on page 254 of the textbook with the order $\mu = \frac{1}{2}$ and with $\lambda = 1$). Therefore a solution to our differential equation is $\sum_{n=0}^{\infty} a_n r^n$, where the a_n are the same coefficients as in the power series expansion for $J_{\frac{1}{2}}(r) = \sum_{n=0}^{\infty} a_n r^{n+\frac{1}{2}}$. In other words, $r^{-\frac{1}{2}} J_{\frac{1}{2}}(r)$ is a solution to our differential equation.

Third solution It happens that $r^{-\frac{1}{2}} J_{\frac{1}{2}}(r)$ is equal to $\sqrt{2/\pi}$ times $r^{-1}\sin r$, as mentioned in the class handout on Bessel functions. Since the differential equation satisfied by trigonometric functions is more familiar than Bessel's equation, we try substituting $R(r) = r^{-1}\sin r$ into the left-hand side of (1):

$$\begin{aligned}
\frac{dR}{dr} &= -r^{-2}\sin r + r^{-1}\cos r \\
\frac{d^2R}{dr^2} &= 2r^{-3}\sin r - 2r^{-2}\cos r - r^{-1}\sin r,
\end{aligned}$$

so $r^2R'' + 2rR' + r^2R = 0$ as claimed.