Complex Variables

Instructions Please write your solutions on your own paper.

These problems should be treated as essay questions. You should explain your reasoning in complete sentences.

1. State the following:

   (a) Cauchy's integral formula;

   (b) the ratio test for convergence of series of complex numbers.

   Solution.

   (a) If C is a simple closed curve, and the function f is analytic on and inside C, then

   $$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz = \begin{cases} f(z_0), & \text{if } z_0 \text{ is inside } C; \\ 0, & \text{if } z_0 \text{ is outside } C; \\ \text{undefined}, & \text{if } z_0 \text{ is on } C. \end{cases}$$

   (b) If $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$ exists and is less than 1, then the series $\sum_{n=1}^{\infty} c_n$ converges. If $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$ exists (or is $+\infty$) and is greater than 1, then the series $\sum_{n=1}^{\infty} c_n$ diverges. If $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$ exists and equals 1, then the ratio test gives no information. If $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$ fails to exist (by oscillation), then the ratio test gives no information.

2. Evaluate the integral $\frac{1}{2\pi i}\int_C \frac{\cos(3z)}{z^3}\,dz$ when C is the unit circle (that is, the set of points z for which $|z| = 1$) oriented in the usual counterclockwise direction.

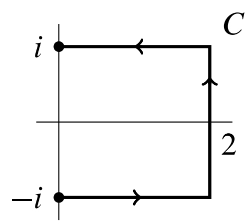
   Solution. Cauchy's formula for derivatives implies that if f is an entire function (or any function that is analytic on the unit disk and its boundary circle), then

   $$\frac{1}{2!}f''(0) = \frac{1}{2\pi i}\int_{|z|=1} \frac{f(z)}{z^3}\,dz.$$

Complex Variables

If $f(z) = \cos(3z)$, then $f''(z) = -9\cos(3z)$, so $f''(0) = -9$. Therefore the original expression to be evaluated equals $-9/2$.

3. Evaluate the integral $\displaystyle\int_C \frac{1}{z^2}\,dz$ when C is the indicated path that goes from the point $-i$ to the point i along three sides of a square.



Solution. The function $-1/z$ is an anti-derivative of $1/z^2$ when $z \neq 0$, so the integral can be evaluated as

$$\left(-\frac{1}{z}\right)\Big|_{z=i} - \left(-\frac{1}{z}\right)\Big|_{z=-i}, \qquad \text{which reduces to } 2i.$$

Alternatively, you could use the path-deformation principle to replace the integration path by a semi-circle in the right-hand half-plane. Parametrizing the new path by setting z equal to $e^{i\theta}$ converts the integral into

$$\int_{-\pi/2}^{\pi/2} \frac{1}{e^{2i\theta}}\, i e^{i\theta}\, d\theta, \qquad \text{or} \qquad \int_{-\pi/2}^{\pi/2} i e^{-i\theta}\, d\theta.$$

The new integral evaluates as $[-e^{-i\theta}]_{-\pi/2}^{\pi/2}$, which again simplifies to $2i$.

4. Determine all values of the real number b for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{b^n + i^n}{(b+i)^n}$ converges.

Solution. Split the series as the sum of the two geometric series

$$\sum_{n=1}^{\infty} \left(\frac{b}{b+i}\right)^n \qquad \text{and} \qquad \sum_{n=1}^{\infty} \left(\frac{i}{b+i}\right)^n.$$

Now b is a real number, so $|b+i|^2 = b^2+1 > b^2$, whence $|b/(b+i)| < 1$. Therefore the first of the two geometric series converges for every value of the real number b. The second of the two geometric series converges when

$|i/(b+i)| < 1$, or $1 < |b+i|$, or $1 < b^2 + 1$. This inequality holds for every real number b except 0.

Consequently, the original series converges as long as the real number b is nonzero. When $b = 0$, however, the original series reduces to $\sum_{n=1}^{\infty} 1$, which diverges to ∞.

Remark By using the formula for the sum of a geometric series, you can show that the given series evaluates to $(b/i) + (i/b)$. It is evident from this expression too that b cannot be allowed to take the value 0.

5. Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2+\cos(n)}{3^n + 4^n} z^n.$$

Solution. The ratio test looks problematic, but the root test shows that the radius of convergence equals the *reciprocal* of

$$\lim_{n\to\infty} \left(\frac{2+\cos(n)}{3^n + 4^n}\right)^{1/n}$$

(if this limit exists).

The intuitive way to compute this limit is to observe that the numerator of the fraction oscillates between definite bounds (namely, between 1 and 3), so the nth root of the numerator should have limit equal to 1. And the denominator is growing roughly like 4^n (since 3^n is much smaller than 4^n when n is large), so its nth root should have limit 4.

To make the argument precise, you could invoke the sandwich theorem (squeeze theorem). Namely, $1 \le 2 + \cos(n) \le 3$, and $4^n < 3^n + 4^n < 4^n + 4^n = 2 \cdot 4^n$, so

$$\frac{1}{2\cdot 4^n} \le \frac{2+\cos(n)}{3^n+4^n} \le \frac{3}{4^n}.$$

Therefore

$$\frac{1}{4}\cdot\frac{1}{2^{1/n}} \le \left(\frac{2+\cos(n)}{3^n+4^n}\right)^{1/n} \le \frac{1}{4}\cdot 3^{1/n}.$$

Since both $2^{1/n} \to 1$ and $3^{1/n} \to 1$ when $n \to \infty$, it follows that

$$\lim_{n\to\infty} \left(\frac{2+\cos(n)}{3^n + 4^n}\right)^{1/n} = \frac{1}{4}.$$

Therefore the radius of convergence of the original power series is equal to 4.

6. Show that if $\operatorname{Im}(z) \neq 0$, then $\sin(z) \neq 0$. In other words, the only values of z in the complex plane for which $\sin(z)$ can be equal to 0 are the values on the real axis where the real sine function is equal to 0.

   Solution. Since $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$, the function $\sin(z)$ is equal to 0 if and only if $e^{iz} - e^{-iz} = 0$, or $e^{iz} = e^{-iz}$. Multiplying by e^{iz} yields the equivalent condition that $e^{2iz} = 1$, or $e^{2ix}e^{-2y} = 1$. Taking the modulus of both sides shows that $e^{-2y} = 1$, whence $y = 0$. Thus the only candidates for complex numbers z that make $\sin(z)$ equal to 0 are points on the x-axis.

   Alternatively, you could write

   $$\sin(z) = \sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y). \qquad (*)$$

   Now $\cosh(y)$ is never equal to 0, and if $y \neq 0$, then $\sinh(y) \neq 0$. Moreover, the functions $\sin(x)$ and $\cos(x)$ are never simultaneously equal to 0, so the real part of (*) and the imaginary part of (*) can never be simultaneously equal to 0 when $y \neq 0$. Thus $\sin(x+iy) \neq 0$ when $y \neq 0$.

Extra credit

Viewing e^z as a transformation from the z-plane to the w-plane, find a region in the z-plane on which the function e^z is a one-to-one transformation onto the upper half-plane (the set of points w having positive imaginary part).

Solution. Write e^z as $e^x e^{iy}$ and observe that $|e^z| = e^x$ and $\arg(e^z) = y$. To cover the upper half of the w-plane, you need the modulus of e^z to run through all positive values and the argument of e^z to vary from 0 to π. Therefore you need x to run over all real values and y to run over all values between 0 and π. In other words, the required region in the z-plane is the horizontal strip $\{\, x+iy : x \in \mathbb{R}$ and $0 < y < \pi \,\}$. It is evident that the function e^z is a one-to-one function on this strip, for two complex numbers are equal if and only if they have the same modulus and the same argument.