# Complex Variables

**Instructions** Please write your solutions on your own paper.

These problems should be treated as essay questions. You should explain your reasoning in complete sentences.

1. State the following:
   a) Euler's formula (relating the exponential and trigonometric functions); and
   b) the power series expansion for $\sin(z)$ (centered at 0).

2. Determine the smallest positive integer $n$ such that
$$\left(\sqrt{3}+i\right)^n = \left(1+i\sqrt{3}\right)^n.$$

3. The set of all complex numbers $z$ such that
$$\operatorname{Re}\left(\frac{1-z}{1+z}\right) = 1$$
can be represented in the plane as a certain curve. What curve is it?
*Caution*: The real part of a quotient is *not* equal to the quotient of the real parts!

4. Let $f(z)$ denote an analytic function with real part $u(x,y)$ and imaginary part $v(x,y)$. Determine $f(z)$ if
$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \text{and} \qquad \frac{\partial v}{\partial x} = 6xy + 1 \qquad \text{and} \qquad f(0) = 0.$$

5. Evaluate the limit
$$\lim_{z\to 0} (\cos z)^{1/z^2}$$
(using the principal branch of the logarithm).

6. The hyperbolic tangent function, tanh, can be defined as follows:
$$\tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}.$$
For which values of the complex variable $z$ is $\tanh(z)$ *not* analytic? In other words, what are the singular points of the hyperbolic tangent function?

## Extra credit

Creatures from the galaxy Mocplex say that a function $u(x,y)$ is *morhanic* if
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0.$$
Are there any nonconstant analytic functions that have morhanic real part?