Complex Variables

Instructions Please write your solutions on your own paper.

These problems should be treated as essay questions. You should explain your reasoning in complete sentences.

1. There are six values of the complex number z for which $z^6 = -1$. Find the solution that has the largest imaginary part.

2. Suppose $v(x, y) = x^3 - 3xy^2$. Find a function $u(x, y)$ such that $u(x, y) + iv(x, y)$ is an analytic function.

3. Find two distinct points z_1 and z_2 in the complex plane such that $\int_C z^2\, dz = 0$, where C is the line segment joining z_1 to z_2. (Notice that the path C is *not* a closed curve!)

4. Does the infinite series
$$\sum_{n=1}^{\infty} \frac{n + i\cos(n)}{in + 3^n}$$
converge? Explain why or why not.

5. If C denotes the unit circle centered at the origin, then which of the two integrals
$$\int_C \frac{\sin(4z)}{\cos(4z)}\, dz \qquad \text{and} \qquad \int_C \frac{\cos(4z)}{\sin(4z)}\, dz$$
has larger modulus? Explain how you know.

6. Determine the largest open annulus in which the Laurent series
$$\cdots + \frac{n^4}{z^n} + \cdots + \frac{3^4}{z^3} + \frac{2^4}{z^2} + \frac{1}{z} + \frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + \cdots + \frac{z^n}{4^n} + \cdots$$
converges.

Extra Credit

When the isosceles right triangle with vertices in the z plane at 0, 1, and i is transformed by the squaring function ($w = z^2$), what is the area of the image region in the w plane? Explain how you know.