# Complex Variables

**Instructions** Please write your solutions on your own paper.

These problems should be treated as essay questions. You should explain your reasoning in complete sentences.

1. State the following:
   a) Liouville's theorem (about entire functions); and
   b) Gauss's mean-value theorem (about functions analytic in a disk).

   **Solution.** Liouville's theorem says that if the range of an entire function is bounded, then the function is a constant function.

   Gauss's mean-value theorem says that if a function $f$ is analytic on and inside a circle of radius $r$ centered at a point $z_0$, then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta;$$

   in other words, the value of the analytic function at the center point is equal to the average of the function around the circle.

2. Evaluate the line integral $\int_C (z - \overline{z})\,dz$ when $C$ is the circle of radius 1 centered at 0 (oriented in the usual counterclockwise direction). [Caution: The answer is not zero!]

   **Solution. Method 1** Parametrize the circle by setting $z$ equal to $e^{i\theta}$, where the angle $\theta$ goes from 0 to $2\pi$. Then $dz = ie^{i\theta}\,d\theta$, and the integral becomes

$$\int_0^{2\pi} (e^{i\theta} - e^{-i\theta})\,ie^{i\theta}\,d\theta \qquad \text{or} \qquad \int_0^{2\pi} (ie^{2i\theta} - i)\,d\theta.$$

   This integral can be evaluated using an antiderivative:

$$\int_0^{2\pi} (ie^{2i\theta} - i)\,d\theta = \left[\tfrac{1}{2}e^{2i\theta} - i\theta\right]_0^{2\pi} = \left(\tfrac{1}{2}e^{4\pi i} - 2\pi i\right) - \left(\tfrac{1}{2}e^{0} - 0\right) = -2\pi i.$$

   **Method 2** The curve $C$ is defined by the property that $1 = |z|^2 = z\overline{z}$, so $\overline{z}$ is equal to $1/z$ on the curve. Therefore

$$\int_C (z - \overline{z})\,dz = \int_C \left(z - \frac{1}{z}\right)dz.$$

   Now $\int_C z\,dz = 0$ by Cauchy's integral theorem, and $\int_C \frac{1}{z}\,dz = 2\pi i$ by Cauchy's integral formula, so the answer to the original problem is $-2\pi i$.

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**Method 3** If $z = x + iy$, then the integral becomes $\int_C 2iy\,(dx + i\,dy)$, which is the sum of $-2\int_C y\,dy$ and $2i\int_C y\,dx$. By Green's theorem, the first integral is equal to 0, and the second integral is equal to $-2i\iint dx\,dy$, where the double integral is taken over the region inside the circle. Thus the answer is $-2i$ times the area of the unit disk, or $-2\pi i$.

3. Determine the values of the complex number $z$ for which the infinite series $\sum_{n=1}^{\infty} e^{nz}$ is convergent.

**Solution.** Since $e^{nz} = (e^z)^n$, the series is a geometric series with ratio $e^z$. Consequently, the series converges when $|e^z| < 1$ and diverges when $|e^z| \geq 1$. If $z = x + iy$, then $|e^z| = |e^x e^{iy}| = e^x$, so the series converges when $x < 0$, that is, when $\operatorname{Re}(z) < 0$. In other words, the convergence region is the open left-hand half-plane.

Applying either the ratio test or the root test shows that the series converges when $|e^z| < 1$ and diverges when $|e^z| > 1$, but neither of those tests will decide the borderline case when $|e^z| = 1$.

**Remark** The convergence region of a power series is always a disk (or, in extreme cases, either the whole plane or a single point). In this problem, however, the series has a different character, being a series in powers of $e^z$ instead of $z$. Accordingly, it is not surprising that the convergence region has a different geometric shape.

4. Evaluate $\displaystyle\int_C \frac{329z + 3\cos(2z)}{z^9}\,dz$ when $C$ is the circle of radius 1 centered at 0 (oriented in the usual counterclockwise direction). [Caution: The answer is not zero!]

**Solution.** By Cauchy's integral formula for derivatives, the integral equals the 8th derivative of the analytic function in the numerator multiplied by $2\pi i/8!$ and evaluated for $z$ equal to 0. The 8th derivative of $329z$ is equal to 0, and the 8th derivative of $3\cos(2z)$ is equal to $3 \cdot 2^8 \cdot \cos(2z)$. Now $\cos(0) = 1$, so the integral equals $3 \cdot 2^8 \cdot 2\pi i/8!$, which simplifies to $4\pi i/105$.

5. Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \left(\frac{3 + 2i}{9^n + i^n}\right) z^n.$$

**Solution.** **Method 1** The root test implies that the radius of convergence is equal to

$$1 \Bigg/ \lim_{n\to\infty} \left|\frac{3+2i}{9^n+i^n}\right|^{1/n} \qquad \text{(if the limit exists).}$$

In the numerator of the limit, $|3+2i|^{1/n} = (\sqrt{9+4})^{1/n} \to 1$ as $n \to \infty$. The term in the denominator can be estimated via the triangle inequality:

$$\tfrac{1}{2}\cdot 9^n < 9^n - 1 \le |9^n + i^n| \le 9^n + 1 < 2\cdot 9^n.$$

Therefore $|9^n + i^n|^{1/n}$ lies between $(1/2)^{1/n}\cdot 9$ and $2^{1/n}\cdot 9$. Since both $(1/2)^{1/n}$ and $2^{1/n}$ have limit equal to 1 when $n\to\infty$, the squeeze theorem from calculus implies that $\lim_{n\to\infty}|9^n+i^n|^{1/n} = 9$. Accordingly, the radius of convergence of the power series is equal to $1/(1/9)$, or 9.

**Method 2** The factor $3+2i$ in the numerator is independent of $n$ and can be factored out of the series. Now

$$\left|\frac{z^n}{9^n+i^n}\right| \le \frac{|z|^n}{9^n-1} = \frac{1}{1-9^{-n}}\cdot\left|\frac{z}{9}\right|^n \le \frac{9}{8}\cdot\left|\frac{z}{9}\right|^n$$

[since the fraction $1/(1-9^{-n})$ takes its largest value when $n=1$]. If $|z|<9$, then

$$\sum_{n=1}^{\infty}\frac{9}{8}\cdot\left|\frac{z}{9}\right|^n$$

is a convergent geometric series, so the comparison test implies that the original series converges when $|z|<9$. Therefore the radius of convergence of the original power series is at least 9.

When $z=9$, the original series becomes

$$(3+2i)\sum_{n=1}^{\infty}\frac{9^n}{9^n+i^n}.$$

This series diverges, because the terms that are being summed do not approach 0. (Indeed, the fraction $9^n/(9^n+i^n)$ approaches the limit 1 as $n$ increases.) Accordingly, the radius of convergence of the original power series cannot be bigger than 9. Combining this observation with the conclusion of the preceding paragraph shows that the radius of convergence is equal to 9.

6. When $C$ is a path in the complex plane from the point $-1$ to the point 1, the value of the line integral

$$\int_C \frac{2}{z^2+1}\,dz$$

depends on the choice of $C$. If $C$ is the straight line segment along the real axis from $-1$ to 1, then the value of this integral is $\pi$. What are the other possible values for this integral (for other choices of the path $C$ joining $-1$ to 1)?

**Solution.** **Method 1** The singularities of the integrand occur when $z$ is equal either to $i$ or to $-i$. If the path $C$ passes *through* one of these two points, then the integral is undefined (more precisely, is a divergent improper integral). If the path $C$ does not pass through either of the singularities, then the path can be converted to a closed path by adding a piece of the real axis from 1 to $-1$. The integral over the extra piece of curve equals $-\pi$, the negative of the integral along the real axis from $-1$ to 1. In other words, the integral over $C$ equals $\pi$ plus the integral over some closed curve.

For a *simple* closed curve, there are four possibilities: the point $i$ is inside the curve and $-i$ is outside; the point $i$ is outside and $-i$ is inside; both $i$ and $-i$ are inside; both $i$ and $-i$ are outside. The partial-fraction decomposition of $2/(z^2+1)$ is $i/(z+i) - i/(z-i)$, so Cauchy's integral formula implies that in the first case, the integral over the closed curve equals $\pm 2\pi$ (the ambiguity in sign reflecting the orientation of the curve, which could be either counterclockwise or clockwise). Similarly, the integral equals either $2\pi$ or $-2\pi$ in the second case. In the third case, the integral equals 0 by cancellation; and in the fourth case, the integral equals 0 by Cauchy's integral theorem.

By the path-deformation principle, the integral over a closed curve that crosses itself is equal to the sum of some number of copies of the integral over a unit circle centered at $i$ and some number of copies of the integral over a unit circle centered at $-i$. In view of the preceding paragraph, the possible values for the integral over a closed curve are precisely the products of $\pi$ with even integers (positive, negative, and zero).

Since the integral over a (nonclosed) path joining $-1$ to 1 differs by $\pi$ from the integral over some closed curve, the possible values for the integral over a path joining $-1$ to 1 are the products of $\pi$ with odd integers.

**Method 2** In a simply connected region containing the points $-1$ and 1 and excluding the points $-i$ and $i$, the integral $\int_{-1}^{z} 1/(w^2+1)\,dw$ makes sense (being independent of the path from $-1$ to $z$) and determines an analytic function of $z$ whose derivative is $1/(z^2+1)$. Thus $\int_{-1}^{1} 2/(z^2+1)\,dz$ equals $2\arctan(1) - 2\arctan(-1)$ for some branch of the arctangent. Since the tangent function is periodic with period $\pi$, different branches of $\arctan(z)$ differ by multiples of $\pi$, and different branches of $2\arctan(z)$ differ by multiples of $2\pi$. The given information that there is some branch of the arctangent for which $2\arctan(1) - 2\arctan(-1) = \pi$ implies that for an arbitrary branch of the arctangent, the expression $2\arctan(1) - 2\arctan(-1)$ equals $\pi$ plus some multiple of $2\pi$. In other words, the only possible values of the original integral are odd multiples of $\pi$. (Additional work is needed to show that every odd multiple of $\pi$ can be realized by choosing a suitable path $C$.)

## Extra credit

Show that if $f$ is an entire function such that $|f(z)| \le 1 + |z|$ for every complex number $z$, then $f$ is a polynomial of degree at most 1.

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**Solution.** **Method 1** The entire function $f$ has a series expansion in powers of $z$ that converges in the whole plane, say

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots .$$

(Moreover $c_n = f^{(n)}(0)/n!$ by Taylor's theorem, but this information is not needed.) Then

$$f(z) - c_0 - c_1 z = z^2(c_2 + c_3 z + c_4 z^2 + \cdots).$$

Let $g(z)$ denote $c_2 + c_3 z + c_4 z^2 + \cdots$. The goal now is to show that $g$ is identically equal to zero, for then $f(z)$ will be equal to the polynomial $c_0 + c_1 z$.

When $z \neq 0$, dividing by $z^2$ shows that

$$g(z) = \frac{f(z) - c_0 - c_1 z}{z^2}.$$

The hypothesis on $f$ implies that $\lim_{|z|\to\infty} |g(z)| = 0$. Accordingly, there is some large radius $R$ such that $|g(z)| < 1$ when $|z| > R$. But $|g(z)|$ is bounded when $|z| \leq R$ because a continuous, real-valued function on a closed, bounded subset of the plane attains a maximal value. Thus $g$ is a bounded entire function, so Liouville's theorem implies that $g$ is a constant function. And that constant value is 0, since $\lim_{|z|\to\infty} |g(z)| = 0$. This deduction completes the argument that $f(z) = c_0 + c_1 z$.

**Method 2** It suffices to show that $f''$, the second derivative of $f$, is identically equal to zero. If $z_0$ is an arbitrary point in the plane, and $R$ is an arbitrary radius, then Cauchy's integral formula for derivatives yields that

$$\begin{aligned} f''(z_0) &= \frac{2!}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^3}\,dz \\ &= \frac{2}{2\pi} \int_0^{2\pi} \frac{e^{-2i\theta} f(z_0 + Re^{i\theta})}{R^2}\,d\theta. \end{aligned}$$

The hypothesis implies that $|e^{-2i\theta} f(z_0 + Re^{i\theta})| \leq 1 + |z_0| + R$, so

$$|f''(z_0)| \leq \frac{2(1 + |z_0| + R)}{R^2}.$$

Since the radius $R$ can be arbitrarily large, and the left-hand side is independent of $R$, it follows that $f''(z_0) = 0$. But the point $z_0$ is arbitrary, so $f''$ is identically equal to zero. Consequently, the function $f$ is a polynomial of degree at most 1.