Examination 1

Instructions Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. Suppose z is a complex number such that the imaginary part of z is equal to 7, and the imaginary part of z^2 is equal to 56. Determine the value of the real part of z.

   Solution. The first piece of information says that $z = a + 7i$ for some real number a. Then $z^2 = (a + 7i)^2 = a^2 - 49 + 14ai$, so the second piece of information implies that $14a = 56$. Therefore the required value of a, the real part of z, is $56/14$, or 4.

2. The set of values of the complex variable z for which

   $$2|z|^2 = |z - i|^2$$

   is a circle in the complex plane. Determine the radius of that circle.

   Solution. Rewrite the equation in terms of the underlying real coordinates x and y by substituting $x + iy$ for z:

   $$2(x^2 + y^2) = x^2 + (y - 1)^2.$$

   Expanding the expression and moving all the variables to the left-hand side shows that

   $$x^2 + y^2 + 2y = 1.$$

   Adding 1 to both sides to complete the square shows that

   $$x^2 + (y + 1)^2 = 2.$$

   Thus the circle has center $(0, -1)$ and radius $\sqrt{2}$.

3. Show that no complex number z exists for which $\tan(z) = i$.
   Hint: Recall that by definition,

   $$\tan(z) = \frac{\sin(z)}{\cos(z)}.$$

   Solution. The standard way to prove nonexistence is to assume that a solution exists and to deduce a contradiction.

   Suppose, then, that there is some complex number z for which

   $$\frac{\sin(z)}{\cos(z)} = i.$$

Multiplying both sides by $\cos(z)$ implies that $\sin(z) = i\cos(z)$, and squaring both sides yields that $\sin^2(z) = -\cos^2(z)$, or $\sin^2(z) + \cos^2(z) = 0$. This equation contradicts the standard trigonometric identity that $\sin^2(z) + \cos^2(z) = 1$. The contradiction means that no solution z exists after all.

You might object that this solution is incomplete, because the solution assumes that the standard trigonometric identity for real variables remains true for complex variables. A way to fill this gap is to use the expressions of the trigonometric functions in terms of exponentials. Namely, if a solution z exists, then

$$\frac{\frac{e^{iz} - e^{-iz}}{2i}}{\frac{e^{iz} + e^{-iz}}{2}} = i, \qquad \text{or} \qquad e^{iz} - e^{-iz} = i^2\left(e^{iz} + e^{-iz}\right).$$

Simplified and expressed in terms of the underlying real variables x and y, the equation says that $2e^{i(x+iy)} = 0$, or $2e^{ix}e^{-y} = 0$. The real exponential function e^{-y} is never equal to 0, so e^{ix} must be equal to 0. By Euler's formula, the absolute value of e^{ix} equals $\sqrt{\cos^2(x) + \sin^2(x)}$, or 1, so the *real* trigonometric identity is contradicted.

Remark. A similar argument shows that also the value $-i$ is missing from the range of the complex tangent function. With more work, you can show that i and $-i$ are the only values missing from the range.

4. a) State the definition of what the complex derivative $f'(0)$ means (in terms of a limit).

Solution. By definition,

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h},$$

where the limit is a two-dimensional limit as $h \to 0$ through complex values.

b) Use the definition to show that if $f(z) = |z|^2$, then the complex derivative $f'(0)$ exists.

Solution. Observe that

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|^2 - 0}{h} = \lim_{h\to 0} \frac{h\overline{h}}{h} = \lim_{h\to 0} \overline{h} = 0,$$

so the derivative exists and equals 0.

Remark. More work shows that $f'(z)$ exists *only* when $z = 0$, so there is no open set on which f is an analytic function.

5. Suppose $f(z)$ is an analytic function, written as usual in the form $u(x, y) + iv(x, y)$ in terms of real functions u and v.

a) State the Cauchy–Riemann equations.

Solution. The Cauchy–Riemann equations say that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

b) Suppose now that $u(x, y) = 2v(x, y)$ for all values of the real variables x and y, and $u(0, 0) = 2$. Determine $f(z)$.

Solution. Combine the given equation with the Cauchy–Riemann equations to see that

$$\frac{\partial u}{\partial x} = 2\frac{\partial v}{\partial x} = -2\frac{\partial u}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = 2\frac{\partial v}{\partial y} = 2\frac{\partial u}{\partial x}.$$

Substituting the value of $\partial u/\partial y$ from the second equation into the first equation reveals that

$$\frac{\partial u}{\partial x} = -2\frac{\partial u}{\partial y} = -4\frac{\partial u}{\partial x}, \qquad \text{whence} \qquad 5\frac{\partial u}{\partial x} = 0, \qquad \text{or} \qquad \frac{\partial u}{\partial x} = 0.$$

Then

$$0 = \frac{\partial u}{\partial x} = -2\frac{\partial u}{\partial y}, \qquad \text{so} \qquad \frac{\partial u}{\partial y} = 0 \quad \text{too.}$$

Since both partial derivatives of the function u are identically equal to 0, the function u must be constant. But $u(0, 0) = 2$, so the constant value is 2. The hypothesis then implies that v is the constant function 1. Thus $f(z)$ is the constant function $2 + i$.

6. Determine a Möbius transformation (fractional linear transformation) that maps the three points -1, 0, and 1 to the image points 0, -1, and 1 (in that order).

Solution. Suppose the transformation sends z to $\dfrac{az + b}{cz + d}$. Since -1 goes to 0, the numerator must be equal to 0 when $z = -1$, that is, $-a + b = 0$, or $a = b$. Since 0 goes to -1, the fraction b/d equals -1, so $b = -d$. Thus $a = b = -d$.

Now 1 goes to 1, so $a + b = c + d$. Combining this equation with the preceding deductions shows that $2a = c - a$, or $c = 3a$. Thus the transformation has the form $\dfrac{az + a}{3az - a}$, or $\dfrac{z + 1}{3z - 1}$.

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Extra Credit

Zed and Zee have a debate about the value of $\lim_{z\to 0} z^z$, where z is a complex variable. Zed says, "0 raised to any power equals 0, so the answer must be 0." Zee says, "Any number raised to the power 0 equals 1, so the answer must be 1."
Do you agree with either Zed or Zee? What do you think about the value of this limit? Explain your reasoning.

Solution. Zed is certainly wrong, yet Zee's response is not entirely satisfactory either. Since the expression 0^0 is an indeterminate expression, the limit cannot be understood by "plugging in the value." A more refined analysis is needed.

The exponent is a complex number, so the expression z^z has to be interpreted as $e^{z\log(z)}$. But $\log(z)$ is not a well-defined function unless a branch cut is made in the plane. Accordingly, the problem does not quite make sense as stated. You cannot take a two-dimensional limit over a full neighborhood of the origin, because z^z is not defined on a whole neighborhood. Moreover, there is not a unique meaning for the symbol z^z.

Nonetheless, you could fix a branch of $\log(z)$ and ask if there is a limit of $e^{z\log(z)}$ as $z \to 0$ within the plane with a branch cut. For example, if you use the principal branch of $\log(z)$, with a cut along the negative part of the real axis, then $\arg(z)$ lies between $-\pi$ and π. Now

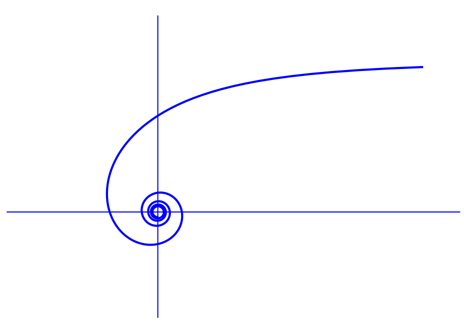
$$z\log(z) = z\ln|z| + zi\arg(z).$$

Since $\arg(z)$ is bounded, the product $zi\arg(z)$ tends to 0 when z tends to 0. And $|z|\ln|z| \to 0$ when $z \to 0$ because $t\ln(t) \to 0$ when $t \to 0$ through positive real numbers. (If you have forgotten this fact from real calculus, rewrite

$$t\ln(t) \qquad \text{as} \qquad \frac{\ln(t)}{1/t}$$

and apply l'Hôpital's rule.) So $e^{z\log(z)} \to e^0 = 1$ for the principal branch of the logarithm when $z \to 0$ within the slit plane.

The same argument shows that $z^z \to 1$ when $z \to 0$ for an arbitrary branch of the logarithm corresponding to a straight-line branch cut. On the other hand, for an exotic branch cut like the spiral shown in the figure, the expression $zi\arg(z)$ fails to have a limit when $z \to 0$ in the cut plane, so z^z too fails to have a limit.



the curve $r\theta = 1, \quad \theta > 0$

Thus the answer is, "It depends." Zed and Zee need to formulate the problem more precisely.